SOLUTIONS
OF THE EXAMPLES IN
A SEQUEL TO
ELEMENTARY GEOMETRY

BY

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NOTICE TO THE READER

THE solutions given in this *Key* are not always applicable, without slight modification, to every possible figure which may be drawn to illustrate the problem concerned. Any reader who from this cause, or any other, finds serious difficulty in the use of this work, is invited to communicate with the author,

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ABBREVIATIONS

| = straight line ; / = divided by.

$\perp$ = perpendicular, is perpendicular to, orthogonal, is orthogonal to ; $\perp^{ly}$ = perpendicularly.

$\parallel$ = parallel, is parallel to ; $\parallel^{m}$ = parallelogram.

$\odot$ = circle ; $\odot^{r}$ = circular ; $\odot$ of s. = circle of similitude.

c. of g. = centre of gravity ; c. of i. rotation = centre of instantaneous rotation ; c. of m. = centre of mass ; c. of s. = centre of similitude ; const. = constant.

h. c. = homothetic centre ; h. r. = homothetic ratio ; h^{c} = harmonic ; h^{ly} = harmonically.

int^{n} = intersection.

max. = maximum ; min. = minimum.

N.P.C. = nine-point circle.

o. c. = orthocentre.

pt. = point ; $proj^{n}$ = projection.

quad. = a figure with four sides and four angles.

r. a. = radical axis ; r. c. = radical centre.

ult^{ly} = ultimately.

vol. = volume.

w. r. to = with respect to.

KEY TO SEQUEL

CHAPTER I

Page 2. Ex. Let the $\perp$ bisector meet AB at L and CD at M; then B, D are the reflexions of A, C in LM. Hence $\angle ACD = BDC = 180° - ABD$ by $\|^s$.

Page 3. Ex. Since $AQ = \frac{1}{2}AP$, the locus of Q is homothetic to l, the locus of P; i.e. is a $|$, $\|$ to l and half-way between A and l.

Page 5. Ex. 1. Let OP cut AB at P''. Then since $PB \parallel OA$, the Δ^s OAP'', PBP'' are similar, $\therefore AP'' : BP'' :: OA : PB :: AP' : BP'$ by hyp. Hence P'' coincides with P'; i.e. O, P, P' are collinear. Also $OP' : OP :: AP' : AB$, a const. ratio. Hence P and P' describe homothetic and, therefore, similar curves.

Ex. 2. Let OO' and PP' meet at S. Then $O'S : OS :: O'P' : OP = k$, say, $\therefore O'S = k.OS$. Hence S is a fixed pt. Also $SP' : SP :: O'P' : OP = k$. Hence $SP' = k.SP$.

Page 6. Ex. Let P' be the reflexion of P in OA and P'' of P' in OB. Then $\angle POA = P'OA$, $P''OB = P'OB$, $\therefore POP' + P'OP'' = 2AOP' + 2P'OB = 2.90° = 180°$. Also $PO = OP' = OP''$. Hence P, O, P'' are collinear and $PO = OP''$.

Page 8. Ex. We shall first prove that the Δ^s $P'CO$ and OAP are similar. Now $P'C : OA :: P'C : BC :: AB : AP$ (from similar Δ^s $P'CB$, BAP) $:: OC : AP$, $\therefore P'C : OC :: OA : AP$. Also the $\angle^s$ $P'CO$, OAP are equal; for $P'CO = P'CB + BCO = BAP + OAB = OAP$. Hence the Δ^s $P'CO$, OAP are similar, $\therefore OP' : OP :: OC : AP$, a const. ratio. Hence $OP' = k.OP$. Again $\angle POP'$ is const.; for $POP' = AOC - P'OC - AOP = AOC - P'OC - CP'O$ (for the similar Δ^s $P'CO$, OAP give $\angle CP'O = AOP$) $= BCX - P'CX$ (producing OC to X) $= BCP'$. Hence, since $OP' = k.OP$ and $\angle POP'$ is const., P and P' describe similar curves.

Page 12. Ex. 1. In the figure on p. 10, let ABC, $A'B'C'$ be two such Δ^s, X, Y, Z being the given pts. and OA, OB the

given $|^s$. Then since AA', BB', CC' concur, the locus of C' (taking ABC as fixed) is the fixed $|$ OC.

Ex. 2. Since ABC, $A'B'C'$ are in perspective, AA', BB', CC' concur. Hence ABC', $A'B'C$ are copolar and $\therefore$ coaxal. Hence $(BC'; B'C)$, $(C'A; CA')$, $(AB; A'B')$ are collinear.

Ex. 3. Since AB, $A'B'$, $A''B''$ concur, $AA'A''$ and $BB'B''$ are copolar and $\therefore$ coaxal; $\therefore$ $(AA'; BB')$, $(A'A''; B'B'')$, $(A''A; B''B)$ are collinear, i.e. the centres of perspective are collinear.

Ex. 4. Consider the Δ^s whose sides are AB, $A'B'$, $A''B''$ and BC, $B'C'$, $B''C''$. Corresponding sides meet at B, B', B'' which are collinear. Hence the Δ^s are coaxal and $\therefore$ copolar. Hence the $|^s$ joining $(AB; A'B')$ to $(BC; B'C')$, $(A'B'; A''B'')$ to $(B'C'; B''C'')$, $(A''B''; AB)$ to $(B''C''; BC)$ concur; i.e. the axes of perspective concur.

CHAPTER II

Page 14. Ex. 1. The Δ^s $OA'B$, $OA'C$ are congruent, $\therefore \angle BOL = COL$, $\therefore \angle BAL = CAL$. Again $\angle LAM = 90°$; $\therefore$ AM is the external bisector of BAC since AL is the internal bisector.

Ex. 2. The Δ^s $BA'L$, $CA'L$ are congruent, $\therefore$ $BL = CL$. Also $\angle PBL = QCL$ (since $BACL$ is cyclic) and $\angle BPL = CQL$ $(=90°)$. Hence the Δ^s LBP, LCQ are congruent.

Now $AP + PB = AB$. Again $AP - PB = AQ - CQ$ (from the congruent Δ^s APL, AQL) $= AC$. Hence $AP = \frac{1}{2}(AB + AC)$ and $PB = \frac{1}{2}(AB - AC)$.

Ex. 3. We have $a = 2R \sin A = 2R \sin A' = a'$, since $A = A'$; so $b = b'$, $c = c'$.

Ex. 4. For brevity take six vertices $A_1, A_2, A_3, A_4, A_5, A_6$. Let P be the point. Let $PA_1 = a_1, \ldots$ and let the $\perp^s$ from P on A_1A_2, A_2A_3, ... be $p_1, p_2, \ldots$. Then, since $bc = 2pR$ in any Δ, we have

$$2Rp_1 = a_1a_2, \quad 2Rp_2 = a_2a_3$$
$$2Rp_3 = a_3a_4, \quad 2Rp_4 = a_4a_5$$
$$2Rp_5 = a_5a_6, \quad 2Rp_6 = a_6a_1.$$

$$\therefore \ 8R^3 p_1 p_3 p_5 = a_1 a_2 a_3 a_4 a_5 a_6 = 8R^3 p_2 p_4 p_6;$$
$$\therefore \ p_1 p_3 p_5 = p_2 p_4 p_6.$$

The same proof holds if we take any even number of sides.

Ex. 5. For brevity take three vertices A_1, A_3, A_5. First take A_2, A_4, A_6 near to A_1, A_3, A_5 on the $\odot$. Then, as above, $p_1 p_3 p_5 = p_2 p_4 p_6$. Now make A_2, A_4, A_6 coincide with A_1, A_3, A_5. Then p_1, the $\perp$ on A_1A_2, becomes the $\perp$ on the tangent at A_1; so p_3, p_5. A similar proof holds for any number of vertices.

Page 17. § 4. Ex. (i) $A'X_1 = A'B - BX_1 = \frac{1}{2}c-(s-c) = A'C - CX = A'X$.

(ii) $XX_2 = BX_2 - BX = s-(s-b) = b$.

Page 17. § 5. Ex. 1. From any pt. A on the first $\odot$, c, draw the tangents to the second $\odot$, i, cutting c at B and C. Let I', r' be the incentre and inradius of ABC. Then I, I' lie on the internal bisector of A. Let AI cut c at L. Now $R^2 - OI^2 = AI.IL$ (as in the text) $= 2Rr$ (by hyp.); also $AI'.I'L = 2Rr'$ (as in the text). Hence $IL : I'L = r\,|\,AI : r'\,|\,AI' = 1$ (by similar Δ^s got by dropping $\perp^s$ from I, I' on AC). Hence I and I' coincide. Hence i is the incircle of ABC; for it has I as centre and touches AB.

Ex. 2. Let the $\odot$ with centre L and radius LB or LI cut IL again at I'. Then $IBI' = 90°$. But $IBI_1 = 90°$. $\therefore$ I' and I_1 coincide, $\therefore$ $LB = LI_1$.

Again $OI_1{}^2 - R^2 = I_1L.I_1A$ (as in the text) $= LB.I_1A$. Also $2Rr_1 = LM.I_1Z_1$. Hence we have to prove that $LB.I_1A = LM.I_1Z_1$, or $LB : LM :: I_1Z_1 : I_1A :: IZ : AI$. Now see the text.

Ex. 3. The square of the tangent $= I_1L.I_1A = 2Rr_1$ by the above.

Ex. 4. If $R = 2r$, $R^2 = 2Rr$, $\therefore$ $OI^2 = 0$. Hence O and I coincide, at S, say. Then, since $IY = IZ$, $SY = SZ$; and $AS = AS$ and $Y = Z = 90°$. Hence $AY = AZ$. But since S is O, $AY = \frac{1}{2}AC$ and $AZ = \frac{1}{2}AB$. Hence $AC = AB$; so $AB = BC$.

Ex. 5. Since $\angle A$ is given, the locus of the pt. A is a $\odot$ on BC containing the angle A. Now L is known, being the bisector of the minor arc of BC. But $LI = LI_1 = LB$ is known. Hence the locus of I and I_1 is the same circle.

Page 19. Ex. 1. $AG = \frac{2}{3}AA' = \frac{2}{3}BB'$ (by hyp.) $= BG$, $\therefore\ \angle ABG = BAG$. Again $BA,\ AA' = AB,\ BB'$, and $\angle BAA' = ABB'$, $\therefore\ BA' = AB'$, $\therefore\ BC = AC$.

Ex. 2. A' is given and A moves on the fixed line BA; also $A'G = \frac{1}{3}A'A$. Hence the locus of G is homothetic to the locus of A and is $\therefore$ a $|$, $\|$ to AB.

Ex. 3. $PC',\ C'B = A'C',\ C'A$ and $\angle PC'B = A'C'A$. $\therefore$ $PB = AA'$. Again $PC' = C'A' = B'C$; hence $PC' =$ and $\|\ B'C$. $\therefore\ PB' = CC'$. And $BB' = BB'$.

Hence $AA' + CC' = PB + PB' > BB'$.

Ex. 4. $GP = AG$ (by hyp.) $= \frac{2}{3}AA'$; so $GC = \frac{2}{3}CC'$. Again $AG : GP :: AB' : B'C$, $\therefore\ PC \parallel GB'$. Hence $PC : GB'$ $:: AC : AB' = 2$, $\therefore\ PC = 2GB' = \frac{2}{3}BB'$. Hence $GP : PC$ $: CG :: AA' : BB' : CC'$.

To construct the Δ, given the lengths x, y, z of the medians, construct the $\Delta\ GCP$ with sides GP, PC, CG equal to $\frac{2}{3}x$, $\frac{2}{3}y$, $\frac{2}{3}z$. Bisect GP at A'; produce CA' to B, so that $A'B = CA'$. Produce PG to A so that $GA = PG$. Then ABC has the given medians. For A' bisects BC; hence AA' is a median. Also $AA' = \frac{3}{2}GP = x$. Again $AG = GP = 2GA'$. Hence G is the centroid. Hence $CC' = \frac{3}{2}CG = z$. Also $BG = CP$ (by the congruent $\Delta^s\ BA'G$, $CA'P$). Hence $BB' = \frac{3}{2}BG = \frac{3}{2}CP = y$.

Page 20. Ex. Since $BA' = A'C$, $\Delta\,BA'A = \Delta\,CA'A$, and $\Delta\,BA'P = \Delta\,CA'P$, $\therefore\ \Delta\,ABP = \Delta\,ACP$.

$\therefore\ \frac{1}{2}AB.BP \sin ABP = \frac{1}{2}AC.CP \sin ACP$.

$\therefore\ AB.BP = AC.CP$, since $ACP = 180° - ABP$.

Page 21. Ex. 1. Now $\angle BHC = FHE = 180° - A$ (since $AFHE$ is cyclic). Hence the locus of H is a $\odot$.

Ex. 2. Let O and H coincide at S. Let AS cut BC at D. Then since S is H, $AD \perp BC$. Again, since S is O and $SD \perp BC$, $\therefore\ D$ bisects BC. Hence $BD,\ DA = CD,\ DA$ and $D = D$, $\therefore\ AB = AC$; so $BC = BA$.

Ex. 3. Let I and H coincide at S. Let AS cut BC at D. Then since S is H, $AD \perp BC$. Again, since S is I, $\angle BAD = CAD$. Hence $\angle^s$ BAD, $ADB = CAD$, ADC and $AD = AD$, $\therefore$ $AB = AC$; so $BC = BA$.

Ex. 4. We have $\Delta = \frac{1}{2} AD . BC = \frac{1}{2} BE . CA$, $\therefore$ $AD . BC = BE . CA$, $\therefore AD : 1/BC :: BE : 1/CA :: CF : 1/AB$ similarly.

Page 22. § 10. Ex. 1. Since BC, CA, AB bisect the $\angle^s$ P, Q, R of the triangle PQR externally, A, B, C are the excentres of PQR. Hence AP bisects $\angle RPQ$ internally. $\therefore$ $AP \perp BC$; hence AP is an altitude of ABC; so BQ, CR.

Ex. 2. Since $AEDB$ is cyclic, $\angle CED = ABC$; and $C = C$. Hence DEC and ABC are similar; i.e. ABC is similar to CDE, and so to AEF and BFD.

Ex. 3. $\angle BAO = 90° - AOC' = 90° - C$ and $\angle CAH = 90° - C$, $\therefore$ $\angle BAO = CAH$. But $\angle BAI = CAI$, $\therefore$ $\angle OAI = HAI$.

Ex. 4. Let AO cut EF at P. Then $\angle APF = PAE + PEA = 90° - B + B = 90°$.

Page 22. § 11. Ex. By the text, the $\odot$ BAC reflects into the $\odot$ BHC in the $|$ BC; hence O reflects into O_1. Hence O, A', O_1 are collinear and $OO_1 = 2 . OA'$; so for O_2, O_3. Hence $O_1O_2O_3$ is homothetic to $A'B'C'$ about O. Hence $O_2O_3 \parallel B'C' \parallel BC$. Hence $OO_1 \perp BC$ and $\therefore$ $\perp O_2O_3$; so $OO_2 \perp O_3O_1$ and $OO_3 \perp O_1O_2$.

Page 23. Ex. 1. $AH^2 + BC^2 = 4A'O^2 + 4A'B^2$

$$= 4OB^2 = 4R^2.$$

Ex. 2. $AB \perp CH$ and DH', $\therefore$ $CH \parallel DH'$. Also $2OC' = CH$ and DH', $\therefore$ $CH = DH'$, $\therefore$ $CD =$ and $\parallel HH'$.

Page 24. Ex. 1. The N.P.C. of AHB passes through X, Y, C' and is $\therefore$ the N.P.C. of ABC; so for BHC, CHA.

Ex. 2. Since $I_1A \perp I_2I_3$ and so on, I is the orthocentre of $I_1I_2I_3$. Hence the $\odot$ ABC is the N.P.C. of $I_1I_2I_3$, and hence bisects I_2I_3 and II_1.

Ex. 3. The locus of A is a $\odot$; hence R is given. Again $A'N = \frac{1}{2}R$ and A' is given; hence the locus of N is a $\odot$.

Ex. 4. Let AA', $B'C'$ meet at S. Then since $AB'A'C'$

is a $\|^m$, S bisects AA' and $B'C'$. Hence the reflexion in S of $AB'C'$ is $A'C'B'$. Hence the N.P.C.s (which pass through S) are reflexions in S and therefore touch.

Ex. 5. We know that $HG = 2GO$ and $HN = NO$. Hence if $OH = 6$ on some scale, $OG = 2$, $GN = 1$, $NH = 3$.

Page 25. Ex. 1. Produce PL to L'' so that $PL'' = PL'$; and so for M, N. Now, since $\angle LPL' = MPM' = NPN'$, $PL' : PL = PM' : PM = PN' : PN = k$, say. $\therefore\ PL'' = PL' = k . PL$; so $PM'' = k . PM$, $PN'' = k . PN$. Hence $L''M''N''$ lie on a $|$, $\|\ LMN$. Again $\angle L'PL'' = L'PL = M'PM = M'PM'' = N'PN''$ similarly; and $PL' = PL''$, $PM' = PM''$, $PN' = PN''$. Hence $L'M'N'$ is $L''M''N''$ turned through the $\angle L''PL'$.

Ex. 2. $\angle NLC = 90° - PLN = 90° - PBA$ (since $PBLN$ is cyclic). Hence $\angle pq = NLC - N'L'C = (90° - PBA) - (90° - QBA) = QBA - PBA = QBP$.

Ex. 3. $PR \perp p$ and $QR \perp q$. But $\angle pq = PAQ$ (by Ex. 2) $= 90°$, $\therefore\ \angle PRQ = 90°$.

Ex. 4. Let PH meet LN at Q. We know that the reflexion c' in BC of the $\odot\ ABC$ is the $\odot\ HBC$. Let P' be the reflexion of P; then $PL = LP'$. Hence $PQ = QH$ if $QL \parallel HP'$, i.e. if $\angle PLQ = PP'H$. But $\angle PLQ = PLN = PBN = PBA = P'BA'$ (by reflexion) $= P'HA' = HP'P$ since $PP' \parallel AA'$.

Or thus, Draw a parabola with P as focus to touch BC, CA. Then since $PL \perp BC$ and $PM \perp CA$, LM is the tangent at the vertex; hence AB also touches since $AB \perp PN$. Also we know that the orthocentre of a Δ circumscribed to a parabola lies on the directrix; which is twice as far from P as LN. Hence $PH = 2 . PQ$.

Ex. 5. $\angle ARP = ABP = NLP$.

Ex. 6. Let the $\odot$s AFE, DCE cut again at P. Draw the $\perp$s PL, PM, PN, PQ on BC, CA, AB, EF. Then N, M, Q are collinear by $\odot\ AFE$ and M, Q, L by $\odot\ DCE$. Hence N, M, Q, L are collinear.

Or, Draw a parabola to touch the four $|$s. Then the feet

of the $\perp^s$ from the focus on the tangents lie on the tangent at the vertex.

Ex. 7. Construct P as in Ex. 6. Then since N, M, L are collinear, the $\odot$ ABC passes through P; and so the $\odot$ BDF.

Or, Draw a parabola to touch the four $\|^s$. Then the $\odot^s$ pass through the focus.

Ex. 8. Drop the $\perp^s$ OX, OY, OZ to DA, DB, DC. Then XY, YZ, ZX are the pedal $|^s$ c, a, b of ADB, BCD, ACD w. r. to O. But O, X, Y, Z are concyclic since $X = Y = Z = 90°$. Hence the projections C', A', B' of O on XY, YZ, ZX are collinear. So A', B', D' are collinear; hence A', B', C', D' are collinear.

Page 27. Ex. 1. Since $PF : PD :: PE : PF$ and $\angle DPF = 180° - DBF$ (since $DPFB$ is cyclic) $= 180° - EAF = FPE$ (since $EPFA$ is cyclic), the Δ^s PFD and PEF are similar, $\therefore$ $\angle PAF = PEF = DFP = DBP$. Hence the $\odot$ APB touches BC at B; and so AC at A. Hence the locus of P is the $\odot$ touching CA, CB at A, B.

Ex. 2. We want to prove that $PL : PM :: PR : PN$. Now $\angle PLM = PCM$ (since $PLCM$ is cyclic) $= PAN = PRN$ (since $PNAR$ is cyclic). So $\angle PML = PNR$. Hence the Δ^s PLM, PRN are similar. Hence $PL : PM :: PR : PN$.

Ex. 3. We are given $PL : PM :: PR : PN$. Also $\angle MPL = 180° - MCL = 180° - NAR = NPR$. Hence the Δ^s MPL and NPR are similar. Hence $\angle PCB = PML = PNR = PAB$. Hence P lies on the $\odot$ ABC; which is $\therefore$ its locus.

End of Chapter II

Ex. 1. (i) $GB + GC > BC$ $\therefore$ $\frac{2}{3}y + \frac{2}{3}z > a$ $\therefore$ $2y + 2z > 3a$.

(ii) $AB' + B'A' > AA'$ $\therefore$ $\frac{1}{2}b + \frac{1}{2}c > x$ $\therefore$ $b + c > 2x$.

(iii) $2y + 2z > 3a$, $2z + 2x > 3b$, $2x + 2y > 3c$ $\therefore$ (adding) $4(x + y + z) > 3(a + b + c)$. Also $b + c > 2x$, $c + a > 2y$, $a + b > 2z$ $\therefore$ $2(a + b + c) > 2(x + y + z)$ $\therefore$ $(x + y + z) : (a + b + c) > \frac{3}{4}$ and < 1.

Ex. 2. We have AA', $A'C = AA'$, $A'B$ and $AC > AB$ $\therefore \angle AA'C > AA'B$ or $GA'C > GA'B$. But GA', $A'C = GA'$, $A'B$ $\therefore$ $GC > GB$ $\therefore$ $\frac{2}{3}z > \frac{2}{3}y$ $\therefore$ $y < z$.

Ex. 3. Let BC be given. Then, since $\Delta = \frac{1}{2} AD . BC$, the length of AD is given. Hence the locus of A is a $\parallel$ to BC. Also $A'G = \frac{1}{3} A'A$. Hence the locus of G is a $\parallel$ to the locus of A, i.e. a $|$, $\parallel$ to BC.

Ex. 4. Since AY bisects $\angle BAC$, Y bisects the arc BC. Also YY' is a diameter since $AX \perp AX'$. Hence $Y'Y \perp BC$, i.e. to $X'X$. Also $XY \perp X'Y'$. Hence Y is the orthocentre.

Ex. 5. We know that $OO_1 = 2 . OA'$ and $OO_2 = 2 . OB'$, $\therefore$ $O_1 O_2 = 2 . A'B' = AB$; so $O_2 O_3 = BC$ and $O_3 O_1 = CA$.

Ex. 6. In the $\Delta\, COA'$, we know $CA' = \frac{1}{2}a$ and $CO = R = a \,|\, 2 \sin A$. Hence OA' is known. Hence $AH = 2 . OA'$ is known. Hence the locus of H is a $\odot$.

Ex. 7. The pedal x of C w. r. to ABP passes through a fixed pt., viz. the projection of C on AB; so the pedal y of D passes through a fixed pt. on AB. Also x, y meet at an $\angle$ equal to DAC. Hence the locus of Q is a $\odot$.

Ex. 8. We know that the four $\odot$s meet again at the second int^n. of the $\odot$s ABF and BCE. Let the $\odot$ BCE cut EF again at K. Then $\angle DAB = BCE = BKF$. Hence the $\odot$ ABF passes through K. Hence the four $\odot$s meet at K.

Ex. 9.
$$\begin{aligned} AD . AH &= AB . AF \text{ (since } HDBF \text{ is cyclic)} \\ &= AE . AC \text{ (since } HDCE \text{ is cyclic).} \end{aligned}$$
$$\begin{aligned} \therefore\ 2\,(AD . AH + BE . BH + CF . CH) &= AB . AF + AE . AC + BC . BD + BF . BA \\ &\quad + CA . CE + CD . CB. \text{ (Similarly)} \\ &= AB\,(AF + FB) + BC\,(BD + DC) + CA\,(CE + EA) \\ &= AB^2 + BC^2 + CA^2. \end{aligned}$$

Ex. 10. See fig. on p. 18. Since $LI = LB = LC$, we see that L is the centre O_1 of the $\odot$ BIC. Hence $O_1 I$ is AI; and $AI \perp O_2 O_3$ because AI is a common chord of the other

$\odot^s$. Hence $O_1I \perp O_2O_3$; so $O_2I \perp O_3O_1$. Hence I is the orthocentre of $O_1O_2O_3$.

Ex. 11. E and F lie on the $\odot$ on BC as diameter. Hence the $\perp$ bisector of EF passes through A' which is the centre of the $\odot$.

Ex. 12. See the fig. on p. 18. Let $A'I$ cut AD at M. Then $AM:AI::LA':LI::LA':LB::IZ:AI$; for $\angle LBA' = LAC = LAB$. Hence $AM = IZ = r$.

Ex. 13. $OL \perp BC$ at A'. Hence OL and BM cut at an angle $90° - CBM = 90° - CAM = B'OA = B$.

Ex. 14. $LI = LB$. Hence we want to prove that $LB^2 = LR \,.\, LA$ or $LB:LR::LA:LB$. Now $\angle LBR = LAC = LAB$ and $\angle BLR = ALB$. Hence the Δ^s LBR and LAB are similar. Hence $LB:LR::LA:LB$.

Ex. 15. See the fig. on p. 18. We want to prove that $AI:AB::AC:AI_1$. Now $\angle IAB = CAI_1$. Also $\angle AIB = 180° - \frac{1}{2}A - \frac{1}{2}B = 90° + \frac{1}{2}C$; and $\angle ACI_1 = ICI_1 + ICA = 90° + \frac{1}{2}C$. Hence $\angle AIB = ACI_1$. Hence the Δ^s AIB and ACI_1 are similar, $\therefore$ $AI:AB::AC:AI_1$.

Ex. 16. Given any three, the fourth is the orthocentre of the three. Also given I, I_1, I_2, I_3, II_1 cuts I_2I_3 at A; and so on.

Ex. 17.
$$\begin{aligned} BO'C &= 2(180° - BPC) \\ &= 360° - 2(PAB + PBA + PAC + PCA) \\ &= 360° - 2A - 2.90° = 180° - 2A \\ &= 180° - BOC. \end{aligned}$$
Hence O, O', B, C are concyclic.

Ex. 18. Let FE meet BC at P. Then $\angle FPC = C - AEF = C - B$. Now $\angle A'ED = EDC - EA'C = A - EA'C = C - B$ if $EA'C = A + B - C = 180° - 2C$. But $\angle EA'C = EFD$ (since A', D, E, F lie on the N.P.C.) $= 90° - EFA + 90° - DFB = 180° - 2C$. Hence $\angle FPC = A'ED$. Also $\angle A'FD = A'ED$ from the N.P.C.

Ex. 19. Draw the $\perp^s$ AL, AM on DC, DB and the $\perp^s$ BX, BY on CD, CA. Let LM, XY cut at P. Then

$\angle MPY = MLX + YXL = DAM + CBY$ (by cyclic quads.) $= 90° - ADB + 90° - BCA = 180° - 2\,ADB = 180° - AOB$. Hence one of the angles between the $|^s = AOB$.

Ex. 20. In Ex. 5 of p. 26, AR is given and $\therefore R$. Thus the $\perp$ from R on BC cuts the $\odot\, ABC$ again at P.

Ex. 21. We know that p passes through P'. We want to prove that $p \perp AQ$. Let LN cut AQ at R. Then $\angle RAN = 180° - BAQ = 180° - PAC = PBC = PNR$ (since $PNLB$ is cyclic) $= 90° - RNA$, $\therefore\ \angle ARN = 90°$.

Ex. 22. By Ex. 6 of p. 26, the Δ^s have the same pedal $|$ w. r. to P. Hence (see the solution of Ex. 4 of p. 26) the pts. Q_1, Q_2, Q_3, Q_4 in which this $|$ meets PH_1, PH_2, PH_3, PH_4 are collinear. But H_1, H_2, H_3, H_4 are such that $PH_1 = 2\,.\,PQ_1$, $PH_2 = 2\,.\,PQ_2$, $PH_3 = 2.PQ_3$, $PH_4 = 2.PQ_4$. Hence H_1, H_2, H_3, H_4 are collinear.

Or thus:—The orthocentres lie on the directrix of the parabola which touches the four lines.

Ex. 23. By Ex. 2 of p. 26, $\angle pq = PRQ$, $\angle qr = QPR$, $\angle rp = RQP$. Hence if $P'Q'R'$ is the Δ formed by p, q, r, we have $P' = P$ or $180° - P$, $Q' = Q$ or $180° - Q$, $R' = R$ or $180° - R$. But $P + Q + R = 180°$. Hence, since $P' + Q' + R' = 180°$, we must have $P' = P$, $Q' = Q$, $R' = R$. For we cannot have $P' = 180° - P$, $Q' = 180° - Q$, $R' = 180° - R$ *or* $P' = 180° - P$, $Q' = 180° - Q$, $R' = R$ or similar cases, *or* $P' = 180 - P$, $Q' = Q$, $R' = R$ or similar cases.

Ex. 24. By Ex. 1 of p. 25, LMN and $L'M'N'$ are $\|$ to the $|^s$ got by turning the pedal $|^s$ of P and P' through a given angle. Hence the angle between them is 90° by Ex. 2 of p. 26.

CHAPTER III

Page 31. Ex. 1. Draw $AE \perp BC$. Then $BE = EC$.

$$\begin{aligned} \therefore\ & BD\,.\,DC + AD^2 - AB^2 \\ &= (BE+ED)(EC-ED) + AE^2 + ED^2 - AE^2 - BE^2 \\ &= (BE+ED)(BE-ED) + ED^2 - BE^2 \\ &= BE^2 - ED^2 + ED^2 - BE^2 = 0. \end{aligned}$$

Ex. 2. $\cos BCA = (BC^2 + AC^2 - AB^2)/2\,BC\,.\,AC$

$\cos CAD = (AD^2 + AC^2 - CD^2)/2\,AD\,.\,AC.$

But $\cos BCA = \cos(180° - CAD) = -\cos CAD$

and $BC = AD$,

$\therefore\ BC^2 + AC^2 - AB^2 = -AD^2 - AC^2 + CD^2$

$\therefore\ AB^2 = BC^2 + AC^2 + AD^2$ since $AC = CD$.

Ex. 3. (i) $\angle RPQ = YPC = 180° - ACZ - BYC = A + a - a = A$, where $a = CZA$; so $Q = B$ and $R = C$.

(ii) $PB : BZ :: \sin a : \sin P$; $QC : CX :: \sin a : \sin Q$; $RA : AY :: \sin a : \sin R$. Also $AB : AX :: \sin a : \sin B$; $BC : BY :: \sin a : \sin C$; $CA : CZ :: \sin a : \sin A$. Now substitute and notice that $A = P$, $B = Q$, $C = R$.

Page 32. Ex. 1. $PR = \frac{1}{2}PQ = \frac{1}{2}(OQ - OP)$.

Ex. 2. $2\,CR = 2\,(OR - OC) = OP + OQ - OA - OB$

and $AP + BQ = OP - OA + OQ - OB = 2\,CR.$

Ex. 3. Let X bisect PQ, $\therefore\ 4\,OX = 2\,OP + 2\,OQ$

$= OB + OC + OA + OC = 2\,OC + OA + OB.$

Let Y bisect CR,

$\therefore\ 4\,OY = 2\,OC + 2\,OR = 2\,OC + OA + OB,$

$\therefore\ OX = OY$, $\therefore\ X$ and Y coincide.

Page 33. Ex. 1. $OR^2 - PR^2 = (OR + PR)(OR - PR)$

$= (OR + RQ)\,OP = OQ\,.\,OP.$

Ex. 2. $2\,CR\,.\,AB - AP\,.\,AQ + BP\,.\,BQ$

$= 2(AR - AC)\,AB - AP\,.\,AQ + (AP - AB)(AQ - AB)$

$= (AP + AQ - AB)\,AB - AP\,.\,AQ + (AP - AB)(AQ - AB)$

$= AP.AB + AQ\,.\,AB - AB^2 - AP\,.\,AQ + AP\,.\,AQ - AP.AB$

$- AB\,.\,AQ + AB^2 = 0.$

Ex. 3. (i) $OA + OB + OC + \ldots - n\,.\,OG$

$= GA - GO + GB - GO + GC - GO + \ldots - n\,.\,OG$

$= GA + GB + GC + \ldots - n\,.\,GO + n\,.\,GO = 0.$

(ii) $OA^2 + OB^2 + OC^2 + \ldots - GA^2 - GB^2 - GC^2 - \ldots - n\,.\,GO^2$

$= (GA - GO)^2 + \ldots - GA^2 - \ldots - n\,.\,GO^2$

$= GA^2 - 2\,GA\,.\,GO + GO^2 + \ldots - GA^2 - \ldots - n\,.\,GO^2$

$= -2\,GO\,(GA + GB + GC + \ldots) + n\,.\,GO^2 - n\,.\,GO^2 = 0.$

Page 34. § 4. Ex. Let O be the centre and r the radius of the $\odot$. Then

$$a^2 . BC + b^2 . CA + c^2 . AB + BC . CA . AB$$
$$= (OA^2 - r^2) BC + (OB^2 - r^2) CA + (OC^2 - r^2) AB + BC . CA . AB$$
$$= OA^2 . BC + OB^2 . CA + OC^2 . AB + BC . CA . AB - r^2 (BC + CA + AB) = 0.$$

Page 34. § 5. Ex. 1. $AM^2 - MB^2 = ME^2 - MB^2$
$= BE^2 = AP^2 - BP^2$.

Ex. 2. We are given $(AO^2 - r^2) - (BO^2 - r^2) = AO^2 - BO^2$. Hence the locus of the centre O is a $| \perp AB$.

Page 35. Ex. 1. Let the $\perp^s$ AX', BY', CZ' on $B'C'$, $C'A'$, $A'B'$ concur; we have to prove that the $\perp^s$ $A'X$, $B'Y$, $C'Z$ on BC, CA, AB concur. Now since AX', BY', CZ' concur,

$$\therefore\ 0 = A'Z'^2 + B'X'^2 + C'Y'^2 - A'Y'^2 - C'X'^2 - B'Z'^2$$
$$= A'Z'^2 - B'Z'^2 + B'X'^2 - C'X'^2 + C'Y'^2 - A'Y'^2$$
$$= A'C^2 - B'C^2 + B'A^2 - C'A^2 + C'B^2 - A'B^2$$
$$= -AC'^2 + BC'^2 - BA'^2 + CA'^2 - CB'^2 + AB'^2$$
$$= -AZ^2 + BZ^2 - BX^2 + CX^2 - CY^2 + AY^2,$$

$$\therefore\ AZ^2 + BX^2 + CY^2 = AY^2 + CX^2 + BZ^2,$$

$\therefore$ $A'X$, $B'Y$, $C'Z$ concur.

Ex. 2. $AZ^2 - BZ^2 + BX^2 - CX^2 + CY^2 - AY^2$

$$= BF^2 - AF^2 + CD^2 - BD^2 + AE^2 - CE^2$$
$$= BC^2 - AC^2 + CA^2 - BA^2 + AB^2 - BC^2 = 0.$$

Ex. 3. Let the $\perp^s$ be I_1X_1, I_2Y_2, I_3Z_3. Then

$$AZ_3^2 - BZ_3^2 + BX_1^2 - CX_1^2 + CY_2^2 - AY_2^2$$
$$= (s-b)^2 - (s-a)^2 + (s-c)^2 - (s-b)^2 + (s-a)^2 - (s-c)^2 = 0.$$

Ex. 4. Let the $\perp^s$ be XX_1, YY_1, ZZ_1. Then

$$AZ_1^2 - BZ_1^2 + BX_1^2 - CX_1^2 + CY_1^2 - AY_1^2$$
$$= AZ^2 - ZZ_1^2 - BZ^2 + ZZ_1^2 + BX^2 - XX_1^2 - CX^2 + XX_1^2 + CY^2 - YY_1^2 - AY^2 + YY_1^2$$
$$= AZ^2 - AY^2 + BX^2 - BZ^2 + CY^2 - CX^2$$
$$= XZ^2 - XY^2 + YX^2 - YZ^2 + ZY^2 - ZX^2 = 0.$$

Page 36. Ex. 1. $AB^2+AC^2=2(A'A^2+A'B^2)$ is given. Hence $A'A$ is given. Also A' is a given pt. Hence the locus of A is a $\odot$.

Ex. 2. Let the $\|^{m}$ be $ABCD$ and O the int^{n} of the diagonals. Then

$$(AB^2+BC^2)+(CD^2+DA^2)=2(BO^2+AO^2)+2(DO^2+AO^2)$$
$$=4AO^2+4BO^2=AC^2+BD^2.$$

Ex. 3. $(AB^2+BC^2)+(CD^2+DA^2)$

$$=2(BE^2+AE^2)+2(DE^2+AE^2)$$
$$=4AE^2+2(BE^2+DE^2)=AC^2+4(EF^2+BF^2)$$
$$=AC^2+4EF^2+4BF^2=AC^2+BD^2+4EF^2.$$

Ex. 4. $AB^2+AC^2=2(AA'^2+BA'^2)$,

$$\therefore\ c^2+b^2=2x^2+\frac{a^2}{2};\ \text{and so on.}$$

$$\therefore\ 4(x^2+y^2+z^2)$$
$$=2c^2+2b^2-a^2+2a^2+2c^2-b^2+2b^2+2a^2-c^2$$
$$=3(a^2+b^2+c^2).$$

Now $3AG=2AA'=2x$,

$$\therefore\ 9(AG^2+BG^2+CG^2)=4(x^2+y^2+z^2)=3(a^2+b^2+c^2).$$

Page 37. § 8. Ex. Take G on BC so that $m\,.\,BG=-n\,.\,GC$. Then

$$m\,.\,BP^2-n\,.\,CP^2=m\,.\,BG^2-n\,.\,CG^2+(m-n)PG^2$$

which is given; hence PG is given. Hence the locus of P is a $\odot$ with centre at G.

Page 37. § 9. Ex. 1. Here

$$m_1PA_1^2+\ldots=m_1\,.\,GA_1^2+\ldots+(m_1+\ldots)PG^2$$

is given. Hence PG is known. Hence the locus of P is a $\odot$.

Ex. 2. $\Sigma(m\,.\,PA^2)$ is least when $\Sigma(m\,.\,GA^2)+PG^2\Sigma m$ is least, i.e. when PG is least, i.e. when P is at G.

Page 38. Ex. 1. Let the $\|^{m}$ be $PQRS$, P being on AB. Then, since $AP:PB::CQ:QB$, if P is the c. of g. of m at A and n at B, Q is the c. of g. of m at C and n at B. Place, then, $2m$ at A, $2n$ at B, $2m$ at C, and $2n$ at D. Now $AS:SD::AP:PB::n:m$ and $CR:RD::AS:SD::n:m$. Hence we can replace the original masses by $m+n$ at P, Q, R, S. Let U be the int^{n} of PR and QS. Then we can

replace $m+n$ at P and $m+n$ at R by $2(m+n)$ at U; and $m+n$ at Q and $m+n$ at S by $2(m+n)$ at U. Hence U is the c. of g. of the original masses. But $2m$ at A and $2m$ at C give $4m$ at E, the centre of AC; and $2n$ at B and $2n$ at D give $4n$ at F, the centre of BD. Hence the c. of g. lies on EF. Hence the locus of U is EF.

Ex. 2. Place masses l^{-1}, m^{-1}, n^{-1} at A, B, C. Then m^{-1} at B and n^{-1} at C give $m^{-1}+n^{-1}$ at X, since $BX.m^{-1} = XC.n^{-1}$. Hence the c. of g., G, of the masses at A, B, C is that of l^{-1} at A and $m^{-1}+n^{-1}$ at X, and hence lies on AX. So G lies on BY, CL. Hence AX, BY, CZ concur. Also $AS.l^{-1} = SX(m^{-1}+n^{-1})$.

Ex. 3. Let E be the c. of g. of masses mk at A and nk at B; then G is the c. of g. of ml at D and nl at C, for $AE:EB :: DG:GC :: n:m$. So H is the c. of g. of mk at A and ml at D, and F is the c. of g. of nk at B and nl at C.

Now place mk at A, nk at B, nl at C, and ml at D. Then mk at A and nk at B give $mk+nk$ at E; so ml at D and nl at C give $ml+nl$ at G. Hence the c. of g. of the original masses lies on EG; and so on HF. Hence EG and HF meet at the c. of g. and ∴ lie in a plane.

Page 40. Ex. 1. In the fig. of the text, take the pt. F on AB so that $AF/FB = a/b$ and E on AC produced so that $AE/CE = c/d$. Let EF cut BC at D. Then

$$(AF/FB).(BD/DC).(CE/EA) = -1,$$
$$\therefore\ (a/b).(BD/DC)\div(-c/d) = -1,$$
$$\therefore\ CD/DB = (a/b)\div(c/d).$$

Ex. 2. Let RQ cut BC at P. Then

$$(CQ/QA)(AR/RB)(BP/PC) = -1,$$
$$\therefore\ CP/BP = (CQ/QA).(AR/RB) = (CQ/QA)^2.$$

Ex. 3. Let the internal and external bisectors of A cut BC at D and D'; and similarly determine E, E' on CA and F, F' on AB. Then

$$(BD/DC)(CE/EA)(AF'/F'B) = (c/b)(a/c)(-b/a) = -1.$$

Hence D, E, F' are collinear. So E, F, D' and F, D, E' and D', E', F' are collinear.

Ex. 4. Let the tangent at A meet BC at D.

Then $BD:DC::(BD/DA):(DC/DA)$,

$::(\sin BAD/\sin DBA):(\sin DAC/\sin DCA)$,

$::(\sin DAF/\sin B):(\sin B/\sin C)$,

$::\sin^2 C:\sin^2 B$.

So for $CE:EA$ and $AF:FB$.

Also D, E, F are outside BC, CA, AB.

Hence $AF.BD.CE = -FB.DC.EA$.

Ex. 5. We know that

$$(BX/CX).(CY/AY).(AZ/BZ) = 1.$$

But $(BX/CX) = \Delta BOX/\Delta COX$

$= OB.OX \sin BOX/OC.OX \sin COX$

$= (OB/OC).(\sin BOX/\sin COX)$.

So for CY/AY and AZ/BZ. Hence

$$(\sin BOX/\sin COX).(\sin COY/\sin AOY).(\sin AOZ/\sin BOZ)$$
$$= (BX/CX).(CY/AY).(AZ/BZ) = 1.$$

Ex. 6. For brevity take 5 pts. It will be seen that the proof applies to any number of pts.

Let LM cut AC at N' and AD at R'.

Then $(AL/BL).(BM/CM).(CN'/AN') = 1$,

$(AN'/CN').(CN/DN).(DR'/AR') = 1$,

$(AR'/DR').(DR/ER).(ES/AS) = 1$.

Multiplying, we get

$(AL/BL).(BM/CM).(CN/DN).(DR/ER)(ES/AS) = 1$.

Ex. 7. Let AC cut LM at X and RN at Y. Then from the ΔABC,

$$AL.BM.CX = -LB.MC.XA,$$

and from the ΔADC,

$$CN.DR.AY = -ND.RA.YC.$$

Multiply and divide by the given relation and we get

$$CX.AY = XA.YC,$$
$$\therefore\ CX:XA::CY:YA.$$

Hence X and Y coincide; i.e. LM, RN meet on AC; so LR, MN meet on BD.

Ex. 8. (1) Suppose the Δ^s are in perspective. Let $B'C'$,

$C'A'$, $A'B'$ cut BC, CA, AB at X, Y, Z. Then from the transversals XB_2C_1, YC_2A_1 and ZA_2B_1 of the ΔABC, we have

$$(BX/XC).(CB_2/B_2A).(AC_1/C_1B) = -1, \quad \text{(i)}$$
$$(CY/YA).(AC_2/C_2B).(BA_1/A_1C) = -1, \quad \text{(ii)}$$
$$(AZ/ZB).(BA_2/A_2C).(CB_1/B_1A) = -1. \quad \text{(iii)}$$

But X, Y, Z are collinear,

$$\therefore \ (BX/XC).(CY/YA).(AZ/ZB) = -1. \quad \text{(iv)}$$

Multiplying (i), (ii), (iii) together, and dividing the result by (iv) we get the given relation.

(2) Suppose the given relation holds. Then (i), (ii), (iii) hold and the given relation holds. Hence (iv) holds, $\therefore$ X, Y, Z are collinear, $\therefore$ the Δ^s are in perspective.

Ex. 9. By Ex. 8, A_2B_1, B_2C_1, C_2A_1 form a Δ in perspective with CAB if the relation of Ex. 8 holds. But in this relation we can interchange A_1 and A_2. Hence A_1B_1, B_2C_1, C_2A_2 form a Δ in perspective with CAB.

Ex. 10. Let the figures PQR... and $P'Q'R'$... be homothetic w. r. to S''; and the figures PQR... and $P''Q''R''$... w. r. to S'. Then $P'Q' \parallel PQ$ and $PQ \parallel P''Q''$; hence $P''Q'' \parallel P'Q'$. Considering P', P'' as fixed pts. and Q', Q'' as variable pts., let $Q'Q''$ cut $P'P''$ at S. Then

$$SP'' : SP' :: Q''P'' : Q'P' :: (Q''P'' : QP) \div (Q'P' : QP)$$

a const. ratio. Hence S is a fixed pt. Also $SQ'' : SQ' :: SP'' : SP'$ is const. Hence Q'', Q' generate homothetic figures about S.

Again, in the $\Delta PP'P''$,

$$S''P \,.\, SP' \,.\, S'P'' = S''P' \,.\, SP'' \,.\, S'P,$$

since $$PQ \,.\, P'Q' \,.\, P''Q'' = P'Q' \,.\, P''Q'' \,.\, PQ.$$

Hence S, S', S'' are collinear.

Page 43. § **14. Ex. 1.**

$$\frac{AF \,.\, BD \,.\, CE}{FB \,.\, DC \,.\, EA} = \frac{(s-b)(s-c)(s-a)}{(s-a)(s-b)(s-c)} = 1.$$

Ex. 2. $SD/AD = \Delta SBD/\Delta ABD = \Delta SCD/\Delta ACD$
$= (\Delta SBD + \Delta SCD)/(\Delta ABD + \Delta ACB)$
$= \Delta BSC/\Delta ABC$;

so SE/BE and SF/CF. Now add.

Ex. 3. We are given that

$$AF \, . \, BD \, . \, CE = FB \, . \, DC \, . \, EA.$$

Also $AF \, . \, AF' \, . \, BD \, . \, BD' \, . \, CE \, . \, CE'$

$$= FB \, . \, F'B \, . \, DC \, . \, D'C \, . \, EA \, . \, E'A$$

since $\quad AF \, . \, AF' = AE \, . \, AE'$ and so on.

Hence $\quad AF' \, . \, BD' \, . \, CE' = F'B \, . \, D'C \, . \, E'A.$

Hence $\quad AD', BE', CF'$ concur.

Ex. 4. We are given that

$$AF \, . \, BD \, . \, CE = FB \, . \, DC \, . \, EA,$$

$$\therefore \frac{BD}{DC} \cdot \frac{CE}{AE} = -\frac{FB}{AF} = \frac{BF}{AF} = 1.$$

Hence F is at infinity. Hence $CS \parallel AB$.

End of Chapter III

Ex. 1. Taking any pt. O on AA' as origin and writing a for OA and so on, we have to prove that

$$(a-p)(a'-p)(w-v)+(b-p)(b'-p)(u-w)$$
$$+(c-p)(c'-p)(v-u) = (a-q)(a'-q)(w-v)$$
$$+(b-q)(b'-q)(u-w)+(c-q)(c'-q)(v-u),$$

or $\quad -p(a+a')(w-v)-p(b+b')(u-w)-p(c+c')(v-u)$
$$+p^2(w-v+u-w+v-u),$$
$$= -q(a+a')(w-v)-q(b+b')(u-w)-q(c+c')(v-u)$$
$$+q^2(w-v+u-w+v-u)$$

or $\quad -p\,2u(w-v)-p\,2v(u-w)-p\,2w(v-u)$
$$= -q\,2u(w-v)-q\,2v(u-w)-q\,2w(v-u),$$

since $\quad a+a' = OA+OA' = 2OU = 2u$ and so on.

And this is true.

Ex. 2. Let the $\perp^s$ be AX, BY, CZ. Then

$$RX^2 - XQ^2 + QZ^2 - ZP^2 + PY^2 - YR^2$$
$$= AR^2 - AQ^2 + CQ^2 - CP^2 + BP^2 - BR^2$$
$$= (s-b)^2 - (s-c)^2 + (s-a)^2 - (s-b)^2 + (s-c)^2 - (s-a)^2 = 0.$$

Ex. 3. $\quad AB^2 + AC^2 = 2(A'A^2 + A'B^2) = 2AA' \, . \, AP,$

if $\quad AA' \, . \, AP - A'A^2 = A'B^2,$

i.e. if $\quad AA'(AP - AA') = A'B^2,$

i.e. if $\quad AA' \, . \, A'P = BA' \, . \, A'C$; which is true.

Ex. 4. See p. 36, Ex. 3. We are given that $EF = 0$. Hence the intn O of AC and BD bisects both AC and BD. Hence AO, $OD = CO$, OB and $\angle AOD = COB$,

$$\therefore \angle OAD = OCB, \therefore AD \parallel BC. \quad \text{So } AB \parallel CD.$$

Ex. 5. $$AB^2 + AC^2 = 2A'A^2 + 2A'B^2,$$

or $$c^2 + b^2 = 2x^2 + 2 \cdot \frac{a^2}{4}, \quad \therefore x^2 = (2b^2 + 2c^2 - a^2)/4.$$

Ex. 6. Let C bisect OA and R bisect PQ. Then

$$OR^2 + AR^2 = 2RC^2 + 2OC^2.$$

Hence $$RC^2 = \tfrac{1}{2}OR^2 + \tfrac{1}{2}AR^2 - \tfrac{1}{4}OA^2$$
$$= \tfrac{1}{2}OQ^2 - \tfrac{1}{2}RQ^2 + \tfrac{1}{2}RQ^2 - \tfrac{1}{4}OA^2;$$

for R is the centre of the $\odot\, PAQ$, since $A = 90°$. Hence RC is const.

Ex. 7. $BX : XC :: BA : AC$. Hence X is the c. of g. of b at B and c at C. Hence

$$b \,.\, AB^2 + c \,.\, AC^2 = b \,.\, XB^2 + c \,.\, XC^2 + (b+c)AX^2.$$

Now $$BX/XC = c/b, \quad \therefore BX/(BX + XC) = c/(b+c),$$
$$\therefore BX = ac/(b+c); \text{ so } CX = ab/(b+c),$$
$$\therefore (b+c)AX^2 = bc^2 + cb^2 - ba^2c^2/(b+c)^2 - ca^2b^2/(b+c)^2$$
$$= bc(b+c) - a^2bc(c+b)/(b+c)^2,$$
$$\therefore AX^2 = bc - a^2bc/(b+c)^2 = bc\left[(b+c)^2 - a^2\right]/(b+c)^2$$
$$= bc(b+c-a)(b+c+a)/(b+c)^2.$$

Ex. 8. Let $BL : LC = p : q$. Then $(p+q)$ at $L =$ (is equivalent to) q at B and p at C; so $(p+q)$ at $M = q$ at C and p at A, and $(p+q)$ at $N = q$ at A and p at B. Hence $p+q$, $p+q$, $p+q$ at L, M, $N = p+q$, $p+q$, $p+q$ at A, B, C; i.e. $3(p+q)$ at the c. of g. of L, M, $N = 3(p+q)$ at the c. of g. of A, B, C; i.e. the two cs of g. coincide.

Ex. 9. Place masses 1, 1, 2 at C, A, B. Then 1 at C and 1 at A and 2 at $B = 2$ at Y and 2 at B. Hence G lies on BY. Also 1 at C and 1 at A and 2 at $B = 1$ at C and 3 at Z. Hence G lies on CZ. Hence G is at P. Hence $CP = 3 \,.\, PZ$.

Ex. 10.

$$\frac{UA}{UD}\cdot\frac{UC}{UB}=\frac{\sin D}{\sin A}\cdot\frac{\sin B}{\sin C} \text{ and } \frac{VA}{VB}\cdot\frac{VC}{VD}=\frac{\sin B}{\sin A}\cdot\frac{\sin D}{\sin C}.$$

Ex. 11. $BX:XC=\Delta BXA:\Delta AXC$

$$= AB\,.\,AX\sin BAX : AX\,.\,AC\sin XAC.$$

So $FP:PE=AF\,.\,AP\sin BAX:AP\,.\,AE\sin XAC,$

$$\therefore\ \frac{BX}{XC}=\frac{AB}{AC}\Big/\frac{AF}{AE} \text{ by division, since } FP=PE,$$

$$\therefore\ \frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}=\frac{AB}{AC}\cdot\frac{BC}{BA}\cdot\frac{CA}{CB}\Big/\frac{AF}{AE}\cdot\frac{BD}{BF}\cdot\frac{CE}{CD}$$

$$=\frac{c}{b}\cdot\frac{a}{c}\cdot\frac{b}{a}\times -1 \text{ (since } D, E, F \text{ are collinear)}$$

$= -1.$ Hence X, Y, Z are collinear.

Ex. 12. $Ab'/Ac'=AA'\sin C'A'A/AA'\sin AA'B'$ and so on.

Hence

$$\frac{Ab'}{Ac'}\cdot\frac{Bc'}{Ba'}\cdot\frac{Ca'}{Cb'}=\frac{\sin C'A'A}{\sin AA'B'}\cdot\frac{\sin A'B'B}{\sin BB'C'}\cdot\frac{\sin B'C'C}{\sin CC'A'}=1,$$

since $A'A$, $B'B$, $C'C$ concur.

Ex. 13. (i) Suppose A', B', C', D' are in the same plane. Then $A'D'$, $B'C'$, BD (being the int^ns of three planes) concur, at X, say. Then $D'A'X$ and the $\Delta\,ADB$ give

$$AA'\,.\,BX\,.\,DD'=A'B\,.\,XD\,.\,D'A.$$

So $$BB'\,.\,CC'\,.\,DX=B'C\,.\,C'D\,.\,XB.$$

Now multiply. Then

$$AA'\,.\,BB'\,.\,CC'\,.\,DD'=A'B\,.\,B'C\,.\,C'D\,.\,D'A.$$

(ii) Suppose

$$AA'\,.\,BB'\,.\,CC'\,.\,DD'=A'B\,.\,B'C\,.\,C'D\,.\,D'A.$$

Then, as in Ex. 7 of p. 41, $D'A'$, $C'B'$ concur.

Hence D', A', C', B' are coplanar.

Ex. 14. $$\frac{PY}{YQ}=\frac{\Delta\,PBY}{\Delta\,YBQ}=\frac{BP\,.\,BY\sin PBY}{BY\,.\,BQ\sin YBQ},$$

$$\therefore\ \frac{\sin PBY}{\sin YBQ}\div\frac{PY}{YQ}=\frac{BQ}{BP}=\frac{\sin P}{\sin Q}; \text{ and so on.}$$

Now multiply up.

Ex. 15. $t_1{}^2 t_2{}^2 t_3{}^2 = DC \,.\, DB \,.\, EA \,.\, EC \,.\, FB \,.\, FA$
$= AF^2 \,.\, BD^2 \,.\, CE^2$ numerically

since $FB \,.\, DC \,.\, EA = AF \,.\, BD \,.\, CE$ numerically.

Ex. 16. $AZ \,.\, BX \,.\, CQ = -ZB \,.\, XC \,.\, QA,$

or $(s-b)(s-c) \,.\, CQ = -(s-a)(s-b) \,.\, QA,$

$\therefore\; CQ/QA = -(s-a)/(s-c)$, and so on.

Now multiply up.

Ex. 17.

$$\frac{AV}{VX} = \frac{\Delta AVB}{\Delta BVX} = \frac{\Delta AVC}{\Delta CVX}$$

$$= \frac{\Delta AVB + \Delta AVC}{\Delta BVC}$$

$$\frac{AZ}{ZB} = \frac{\Delta AVC}{\Delta BVC}, \quad \frac{AY}{YC} = \frac{\Delta AVB}{\Delta BVC}.$$

Now substitute.

Ex. 18. AX, BY, CZ concur,

if
$$\frac{\sin BAX}{\sin XAC} \cdot \frac{\sin ACZ}{\sin ZCB} \cdot \frac{\sin CBY}{\sin YBA} = 1,$$

or briefly if $\Pi \sin BAX/\sin XAC = 1.$

Now, since PX, QY, RZ concur, $\therefore\; \Pi\, RX/XQ = 1,$

$\therefore\; \Pi\, \Delta RAX/\Delta XAQ = 1,$

$\therefore\; \Pi\, AR \,.\, AX \sin BAX/AX \,.\, AQ \sin XAC = 1,$

$\therefore\; \Pi \sin BAX/\sin XAC = \Pi\, AQ/AR = 1,$

since AP, BQ, CR concur.

Ex. 19. We are given that

$$BX \,.\, CY \,.\, AZ = -XC \,.\, YA \,.\, ZB,$$

or $s(s-a)(s-a) = (s-a)(s-c)(s-b),$

or $s^2 - as = s^2 - s(b+c) + bc,$

or $(b+c-a)(b+c+a) = 2bc,$

or $b^2 + 2bc + c^2 - a^2 = 2bc,$

$\therefore\; b^2 + c^2 = a^2.$

Ex. 20. We are given that

$$BX \,.\, CY \,.\, AZ = XC \,.\, YA \,.\, ZB.$$

But since BC and XX' have the same centre, $BX = X'C$; and so on.

Hence $X'C \,.\, Y'A \,.\, Z'B = BX' \,.\, CY' \,.\, AZ'.$

Hence AX', BY', CZ' concur.

CHAPTER IV

Page 47. Ex. 1. $\angle MNA + P'AN = MPA + PAM = 90°$. Hence $MN \perp AP'$.

Ex. 2. $\angle BAO = 90° - C = CAH$.

Page 48. § 2. Ex. 1. We must have $\alpha = \alpha'$, $\beta = \beta'$, $\gamma = \gamma'$ in $\alpha\alpha' = \beta\beta' = \gamma\gamma'$, $\therefore\ \alpha = \pm\beta = \pm\gamma$. Hence I, I_1, I_2, I_3 are the only such pts.

Ex. 2. For if P is any pt. on BC, then $\angle BCP = ACA\,(= 0)$ and $CBP = ABA$.

Ex. 3. With P and P' as foci describe a conic to touch BC. Then since $PL\,.\,P'L' = PM\,.\,P'M' = PN\,.\,P'N'$, the conic also touches CA and AB. Hence L, L', M, M', N, N' lie on the auxiliary ⊙ of the conic.

Or. $AM\,.\,AM' = AP\cos PAC\,.\,AP'\cos P'AC$ $= AP\cos P'AB\,.\,AP'\cos PAB = AP\cos PAB\,.\,AP'\cos P'AB$ $= AN\,.\,AN'$. Hence M, M', N, N' lie on a ⊙ whose centre bisects PP', at V, say. Hence $VM = VM' = VN = VN'$ $=$ (also, similarly) $VL = VL'$. Hence L, L', M, M', N, N' are concyclic.

Page 48. § 3. Ex. 1. We know that

$$\Delta AGB = \Delta BGC = \Delta CGA,\ \therefore\ \tfrac{1}{2}a\alpha = \tfrac{1}{2}b\beta = \tfrac{1}{2}c\gamma,$$
$$\therefore\ \alpha' : \beta' : \gamma' :: \alpha^{-1} : \beta^{-1} : \gamma^{-1} :: a : b : c.$$

Ex. 2. $BP : PC :: \Delta BKA : \Delta CKA :: \tfrac{1}{2}c\gamma' : \tfrac{1}{2}b\beta' :: c\,.\,c : b\,.\,b$ for $\gamma' : \beta' :: c : b$.

Page 49. § 4. Ex. 1. $\angle AEF = ABC$.

Ex. 2. (i) Let the intercepts XY and $X'Y'$ between $AX'XB$ and $AYY'C$ be ∥ the isogonal |ˢ AP and AP'. (In the figure, take AP and AP' outside BAC for convenience.) Then $\angle AXY = PAB = P'AC = AY'X'$. Hence XY and $X'Y'$ are antiparallel. (iii) is proved similarly.

(ii) Let XY and $X'Y'$ be $\perp$ AP and AP'. Then $\angle AYX$ $= 90° - PAC = 90° - P'AB = AX'Y'$. Hence XY and $X'Y'$ are antiparallel. (iv) is proved similarly.

Ex. 3. Let AT be the tangent at A. Then $\angle TAC = ABC$.

Page 49. § 5. Ex. Let the ∥ through T to the tangent

at A cut AB at P and AC at Q; and let the tangent cut TB at X and TC at Y. Then $\angle QPA = PAX = BCA$. Hence PQ and BC are antiparallel.

Again $\angle TPB = BAX = ABX = TBP$, $\therefore$ $TB = TP$; so $TC = TQ$. But $TB = TC$, $\therefore$ $TP = TQ$. Hence AT bisects an antiparallel to BC and is $\therefore$ a symmedian.

Page 50. Ex. Let $PA = x$, $PB = y$ and $AB = c$. Then $u = lx + my$ has to be greatest, given $x^2 + y^2 = c^2$. Now

$$(lx + my)^2 + (ly - mx)^2 = (l^2 + m^2)(x^2 + y^2),$$

$\therefore$ $u^2 = (l^2 + m^2)c^2 - (ly - mx)^2$ is greatest when $ly - mx = 0$, i.e. when $x : y :: l : m$. Hence the angle PAB can be constructed, since $x : y$ is known.

Page 51. § 7. Ex. 1. First, with the figure of § 7, suppose BC given and also the angle BCA. Then the locus of Ω is the $\odot$ touching CA at C and passing through B. So in the given case, the locus of Ω' is the $\odot$ touching BA at B and passing through C.

Ex. 2. Let BR cut the $\odot$ at X. Then $\angle ACX = BAX = ARB = CBX$ by $\|^s$.

Ex. 3. $\angle B\Omega C = 180° - \omega - (C - \omega) = 180° - C$.

End of Chapter IV

Ex. 1. Let D, E, F be the collinear pts. and AD', BE', CF' the isogonal conjugates. Then $\sin ACF . \sin BAD . \sin CBE = -\sin FCB . \sin DAC . \sin EBA$. But $ACF = F'CB$, and so on. Hence $\sin F'CB . \sin D'AC . \sin E'BA = -\sin ACF' . \sin BAD' . \sin CBE'$. Hence D', E', F' are collinear.

Ex. 2. Let $\angle BAP = QAC = D$. Then

$$\frac{AP}{BP} \cdot \frac{AP}{PC} = \frac{AQ}{BQ} \cdot \frac{AQ}{QC},$$

if

$$\frac{\sin B}{\sin D} \cdot \frac{\sin C}{\sin (A - D)} = \frac{\sin B}{\sin (A - D)} \cdot \frac{\sin C}{\sin D}.$$

Ex. 3. Let P' be the pt. at infinity on AQ. It is sufficient to prove that $\angle ABP' = CBP$. Now $BP' \parallel AQ$. Hence $ABP' = BAQ = CAP = CBP$.

Ex. 4. $PL/BP = \sin PBL = \sin P'BN' = P'N'/P'B$,
$\therefore$ $BP/BP' = PL/P'N' = a/\gamma' \propto a\gamma \propto \beta^{-1} \propto (PM)^{-1}$.

Ex. 5. If B' is (a', β', γ'), the equation of BB' is

$$\gamma/a = \gamma'/a' = c/a, \text{ since } a' = -a, \gamma' = -c.$$

Hence AA', BB', CC' meet at the pt. $a:\beta:\gamma::a:b:c$.

Ex. 6. Let P, (a', β', γ'), bisect AD. Then

$$\gamma'/a' = \tfrac{1}{2}AD\sin BAD/\tfrac{1}{2}AD = \cos B = c/a.$$

So $$\beta'/a' = b/a, \therefore a':\beta':\gamma'::a:b:c.$$

Ex. 7. (i) Since ABC, PQR are inscribed in the same $\odot$, they are congruent if similar. Now $\angle QRP = QRC + PRC = QBC + PAC = BAP + PAC = A$; so $Q = C$, $P = B$. Hence the Δ^s are similar.

(ii) $\angle QR\Omega = QBC = BAP = PQ\Omega$; and so on. Hence Ω is a Brocard pt. of PQR.

Ex. 8. For clearness let P, Q, R, R' be called B, A, C, C'. Then Ω of ABC lies on the $\odot$ ACC', since this touches BA at A and passes through C. Also Ω is a Brocard pt. of ABC'; for $\angle AC'\Omega = AC\Omega = \omega$, $C'B\Omega = CB\Omega = \omega$, $BA\Omega = \omega$. Also ω is the new Brocard angle.

Ex. 9.
$$A\Omega = AC\sin\omega/\sin[180^\circ - \omega - (A - \omega)]$$
$$= b\sin\omega/\sin A; \text{ and so on.}$$
and
$$A\Omega' = AB\sin\omega/\sin[180^\circ - \omega - (A - \omega)]$$
$$= c\sin\omega/\sin A; \text{ and so on.}$$

Now substitute.

CHAPTER V

Page 54. Ex. Since (BC, XX') is h^c,

$$BX/XC = -BX'/X'C; \text{ and so on.}$$

Since AX, BY, CZ concur,

$$BX.CY.AZ = XC.YA.ZB,$$
$$\therefore BX'.CY'.AZ' = -X'C.Y'A.Z'B.$$

Hence X', Y', Z' are collinear.

Page 56. Ex. 1. $OC.OD = OA^2$. Hence OC and OD have the same sign.

Ex. 2. $AC:AD::OC:AO$ if $AC.AO = AD.OC$, i.e. if $(c-a)(-a) = (d-a)c$, where $c = OC$, and so on, i.e. if $-ca+a^2 = dc-ac$, i.e. if $a^2 = cd$.

Ex. 3. $AB^2+CD^2 = 4UV^2$ if $(b-a)^2+(d-c)^2 = 4v^2$,

where $\qquad b = UB = -UA = -a$

and $\quad 4v^2 = 4UV^2 = (UC+UD)^2 = (c+d)^2 = c^2+2cd+d^2$,

i.e. if $\qquad 4a^2+d^2-2dc+c^2 = c^2+2cd+d^2$,

i.e. if $\qquad a^2 = cd$.

Ex. 4. Taking B as origin,

$$AB.CD+2AD.BC = -a(d-c)+2(d-a)c$$
$$= -ad+ac+2cd-2ac = 2cd-ad-ac = 0;$$

for $\qquad 2/a = 1/c+1/d$.

Ex. 5. $CA.CB+DA.DB-CD^2$

$$= (a-c)(-c)+(a-d)(-d)-(d-c)^2$$
$$= -ac+c^2-ad+d^2-d^2+2dc-c^2$$
$$= 2cd-ad-ac = 0.$$

Ex. 6. $PA.BC+PB.AD+PC.DB+PD.CA$

$$= (a-p)c+(-p)(d-a)+(c-p)(-d)+(d-p)(a-c)$$
$$= ac-pc-pd+pa-cd+pd+da-dc-pa+pc$$
$$= ad+ac-2cd = 0.$$

Page 58. § 3. Ex. 1. A section of the pencil $A'(C'AB'C)$ is $C'AIB$ where I is at infinity on BA; and this is h^c^ since C' bisects BA.

Ex. 2. Let VD cut $A'B'$ at E'. Then $(A'B'C'E')$ is h^c^, being a section of the h^c^ pencil $V(ABCD)$; and $(A'B'C'D')$ is h^c^, ∴ E' coincides with D'.

Ex. 3. Let VD and $V'D'$ meet AB at E and E'. Then $(ABCE)$ and $(ABCE')$ are h^c^, ∴ E and E' coincide.

Ex. 4. (i) Draw the ⊥^s^ from C and D. Then

$$p_1/p_2 = VC\sin AVC/VC\sin CVB,$$

and $\qquad p_3/p_4 = -VD\sin AVD/VD\sin DVB$

$\qquad$ (−, since p_4 is − if p_2 is +),

$$\therefore\ p_1/p_2 = p_3/p_4.$$

(ii) If $p_1/p_2 = p_3/p_4$, then, as above,

$$\sin AVC/\sin CVB = -\sin AVD/\sin DVB,$$
$$\therefore\ V(AB, CD) \text{ is h}^c.$$

Page 58. § **4.** **Ex. 1.** Let VE be the other bisector. Then $V(AB, CE)$ is h^c; and also $V(AB, CD)$. Hence VD coincides with VE.

Ex. 2. We know that AI, AI_2 bisect $\angle BAC$. Also $\angle OAB = 90° - C = HAC$; hence AI and AI_2 also bisect $\angle OAH$.

Page 59. **Ex. 1.** Let the segments be PP' and QQ'. Through any pt. V draw the ⊙s VPP' and VQQ' cutting again at V'. Let VV' cut PP' at O. Draw OT touching either ⊙. With O as centre and OT as radius draw a ⊙ cutting PQ' at E and F. Then E, F are h^c with both PP' and QQ'. For O bisects EF; and $OE^2 = OT^2 = OV.OV' = OP.OP'$, $\therefore$ (EF, PP') is h^c. So (EF', QQ') is h^c. O is the centre and E, F the double pts. of the involution determined by PP', QQ'.

Ex. 2. Take the section $PP'QQ'$ of the pencil and construct E, F as in Ex. 1. Then $V(EF, PP')$ and $V(EF, QQ')$ are h^c.

Ex. 3. For clearness write U for p and V for q. Then

$$\begin{aligned} QP.QP' - 2QO.VU &= (p-q)(p'-q) + 2q(u-v) \\ &= pp' - pq - p'q + q^2 + q(p + p' - q - q') \\ &= pp' - pq - p'q + q^2 + pq + p'q - q^2 - qq' = 0, \end{aligned}$$

since $\quad pp' = OP.OP' = OQ.OQ' = qq'$.

Ex. 4. $PA.PB - PC.PD + 2UV.PO$

$$\begin{aligned} &= (a-p)(b-p) - (c-p)(d-p) + 2(v-u)(-p) \\ &= ab - ap - pb + p^2 - cd + cp + pd - p^2 + (c + d - a - b)(-p) \\ &= ab - ap - bp - cd + cp + dp - cp - dp + ap + bp \\ &= ab - cd = 0, \end{aligned}$$

for $\quad ab = OA.OB = OC.OD = cd$.

Page 61. **Ex. 1.** (i) (BC, PX) is h^c because BC is a diagonal of the quadrilateral $ARSQA$; so (CA, QY) and (AB, RZ).

(ii) Now use p. 54. Ex.

(iii) AX, CZ, BQ concur if

$$AZ \,.\, BX \,.\, CQ = ZB \,.\, XC \,.\, QA.$$

But $\qquad AR \,.\, BP \,.\, CQ = RB \,.\, PC \,.\, QA$

and $\quad AZ/ZB = -AR/RB$ and $BX/XC = -BP/PC$.

Ex. 2. Let PQ meet AR at U and BC at I (at infinity). Then the diagonal PQ of $APRQA$ is divided h^{ly} by U, I. Hence U bisects PQ.

Ex. 3. Take L on BC and M on CA so that (BC, XL) and (CA, YM) are h^c. Now project LM to infinity, taking any vertex of proj^n. Then, in the new figure, $(B'C', X'L')$ is h^c and L' is at infinity; hence X' bisects $B'C'$; so Y' bisects $C'A'$. Hence S' is the centroid of $A'B'C'$.

Ex. 4. Take I on LM and J on PQ so that (LN, MI) and (PR, QJ) are h^c. Now project IJ to infinity.

Page 62. Ex. 1. In the figure of § 8, draw the | through $W \parallel UV$, cutting BC, BA, CD, AD at P, Q, R, S. Then, since $V(UW, AC)$ is h^c, (IW, SP) is h^c. Also I is at infinity; hence W bisects SP. So W bisects QR.

Ex. 2. Let the $\parallel$ cut AU, VU at Y, X. Then, since $U(VW, AC)$ is h^c, (XW, YI) is h^c. Hence Y bisects XW.

End of Chapter V

Ex. 1.
$$2\frac{PB}{AB} = \frac{PC}{AC} + \frac{PD}{AD},$$

if
$$2\,\frac{b-p}{b} = \frac{c-p}{c} + \frac{d-p}{d}, \text{ (A being origin)}$$

i.e. if
$$\frac{2}{b} = \frac{1}{c} + \frac{1}{d}.$$

Ex. 2. Taking U as origin

$$\begin{aligned} & PA \,.\, PB + PC \,.\, PD - 2PU \,.\, PV \\ &= (a-p)(b-p) + (c-p)(d-p) + p\,(2v - 2p) \\ &= ab - ap - bp + p^2 + cd - cp - dp + p^2 + p\,(c+d) - 2p^2 \\ &= -a^2 + cd = -UA^2 + UC \,.\, UD = 0, \end{aligned}$$

for $\; b = UB = -UA = -a$.

Ex. 3. Let BB', CD' meet at V; and let DV cut AB' at E'. Then $(AB', E'D')$ is h^c, since it is a section of the h^c pencil $V(AB, DC)$. But $(AB', C'D')$ is h^c; hence E' coincides with C'.

Ex. 4. Take the centre as origin. Then

$$aa' = bb' = \dots = k \text{ and } a + a' = 2u,\ b + b' = 2v,\ c + c' = 2w,$$

$$\begin{aligned} \therefore\ & PA \,.\, PA' \,.\, VW + PB \,.\, PB' \,.\, WU + PC \,.\, PC' \,.\, UV \\ &= (a-p)(a'-p)(w-v) + (b-p)(b'-p)(u-w) \\ &\qquad + (c-p)(c'-p)(v-u) \\ &= k(w-v+u-w+v-u) \\ &\qquad -p[2u(w-v) + 2v(u-w) + 2w(v-u)] \\ &\qquad + p^2[w-v+u-w+v-u] = 0. \end{aligned}$$

Ex. 5. Project DE to infinity. Then l' and m' are $\parallel$. Also $G'F' \parallel A'B'$ and $H'F' \parallel A'C'$. Hence $G'H' = G'A' + A'H' = F'B' + C'F' = C'B'$. Hence $B'H' \parallel C'G'$. Hence BH, CG meet on n.

Ex. 6. Let the $\parallel^s$ DB, $D'B'$ cut AC at E, E'. Then E, E' bisect DB, $D'B'$. Hence (DB, EI) and $(D'B', E'I)$ are h^c, I being at infinity. Hence EE' is the polar of I w. r. to AB and AD and $\therefore$ coincides with the | joining A to the int^n of DB' and $D'B$, i.e. this int^n lies on AC'.

Ex. 7. Let $B'C'$ cut BC at X'. Then

$$AB' \,.\, CX' \,.\, BC' = -B'C \,.\, X'B \,.\, C'A.$$

Let $B''C''$ cut BC at X''. Then

$$AB'' \,.\, CX'' \,.\, BC'' = -B''C \,.\, X''B \,.\, C''A.$$

But $\quad AB'/B'C = -AB''/B''C$

and $\quad BC'/C'A = -BC''/C''A.$

Hence $\quad CX'/X'B = CX''/X''B.$

Hence X' and X'' coincide. So for the rest.

Ex. 8. Let AO cut BC at D. Then (BC, DA') is h^c, since $A(C'B', OA')$ is h^c. Hence A' is known.

Ex. 9. Let AC and BD cut at O. Then

$$OX \,.\, OY = OB^2 = OC^2 = OP \,.\, OQ.$$

Hence P, Q, X, Y are concyclic.

Ex. 10. For clearness let O be called X. Take the centre R of AB as origin; then $b = -a$ and $a^2 = cd$. Also

$$2p = b+c,\ 2p' = a+d,\ 2q = c+a,\ 2q' = b+d,$$
$$o = a+b,\ 2r' = c+d,$$
$$\therefore\ 8XR\,.\,XR' - 4XP\,.\,XP' - 4XQ\,.\,XQ'$$
$$= 2(-2x)(2r'-2x) - (2p-2x)(2p'-2x) - (2q-2x)(2q'-2x)$$
$$= -4x(c+d-2x) - (b+c-2x)(a+d-2x) - (c+a-2x)(b+d-2x)$$
$$= 2x(-2c-2d+a+d+b+c+c+a+b+d)$$
$$-(ba+bd+ca+cd+cb+cd+ab+ad)$$
$$= a^2+ad-ac-cd+ac-cd+a^2-ad$$
$$= 2a^2-2cd = 0.$$

Ex. 11. Let the tangent at P cut AB at O. Then

$$OP^2 = OA\,.\,OB = OC\,.\,OD.$$

Hence O is a fixed pt. and OP is a fixed length. Hence the locus of P is a ⊙.

CHAPTER VI

Page 65. Ex. 1. Since $OB:OP::OP:OB'$, the Δ^s OBP, OPB' are similar. Hence $OB:OP::PB:PB'$.

Ex. 2. $$\frac{PB}{PB'} = \frac{OB}{OP} = \frac{OP}{OB'} = \sqrt{\frac{OB\,.\,OP}{OP\,.\,OB'}}.$$

Ex. 3. (i) On the same radius OA. Then $\angle APB = APB'$ and $APC = APC'$, $\therefore$ $BPC = C'PB'$.

(ii) On opposite radii. Produce $C'P$ to D'. Then PA bisects $\angle D'PC$. Hence $\angle APC = APD'$ and $APB = APB'$, $\therefore$ $BPC = B'PD' = 180° - C'PB'$.

Ex. 4. Let AB, CD be the two segments. (i) Suppose AB and CD are outside one another. Let P be a pt. such that $\angle APB = CPD$. Then the angles APD and BPC have the same bisectors; let these cut AD at E, F. Then E, F are h^c with AD and BC; and hence are known. Also $\angle EPF = 90°$. Hence P lies on the ⊙ on EF as diameter.

Conversely, if P is any pt. on this $\odot$ (centre O), then since $OA \, . \, OD = OB \, . \, OC = OE^2$, A, D and B, C are pairs of inverse pts. Hence $\angle APB = CPD$.

(ii) Let AB, CD overlap. Then, since $\angle APB = CPD$, $\therefore$ $APC = BPD$. Also AC and BD are outside one another. Hence case (ii) is reduced to case (i).

(iii) Let CD be inside AB. Produce AP to A'. Then we must have $\angle A'PB = CPD$. The solution now proceeds as in case (i), PE, PF being the bisectors of the angles $A'PC$, BPD, and E, F h^c with both A, C and B, D.

Page 67. § 3. Ex. Let p and q cut at X and Y. Then

$$BX/XC = BA/AC \text{ and } CX/XA = CB/BA$$
$$\therefore \; BX/XA = BA/AC \div CB/BA = BC/CA.$$

Hence X (and so Y) lies on r.

Page 67. § 4. Ex. 1. Draw the radius OC of the $\odot \, ABC$. Then $\angle ACO = CAB = CDE$. Hence OC touches $\odot \, CDE$.

Ex. 2. Let the tangents at one int^n A meet the $\odot^s$ at P and Q. Join P and Q to the other int^n B. Then, since AP touches one $\odot$, it is a diameter of the other. Hence $ABP = 90°$; so $ABQ = 90°$.

Ex. 3. Draw (outwardly) the tangents AP and AQ at A to the $\odot^s$ ACB and ADB. Then $\angle DBC = DBA + ABC$ $= QAD + PAC = 180° - PAQ$. Hence $PAQ = 90°$ when $DBC = 90°$.

Ex. 4. Let a and b be the radii. Then $a^2 + b^2 = AH \, . \, AD$ $+ BH \, . \, BE = AF \, . \, AB + BF \, . \, BA = AB^2 = d^2$.

Page 69. Ex. See the figure on p. 60. $(AA', \beta\gamma)$ is h^c. Hence β, γ are inverse w. r. to the $\odot$ a on AA' as diameter. Hence the $\odot \, a\beta\gamma$ is $\perp$ $\odot$ a; and so to b, c.

Page 70. Ex. (i) Since O bisects AA', $\therefore$ $PO = \frac{1}{2}(PA + PA')$.

(ii) $PT^2 = PA \, . \, PA'$.

(iii) Since (PR, AA') is h^c, $2/PR = 1/PA + 1/PA'$.

Page 71. Ex. 1. Let PQ cut the polar of A at R. Then $\angle ABR = 90°$ and $B(QP, AR)$ is h^c. Hence AB bisects $\angle PBQ$.

Ex. 2. Let a $\parallel$ to PS through V cut PQ at N, $\therefore$ $VN \perp PQ$. Again the h^c pts. of the quadrangle $PSTQ$ are U, V and the pt. I at infinity on PS. Hence $V(PQ, UI)$ is h^c, $\therefore$ (PQ, UN) is h^c. Hence the polar of U passes through N; and it is $\perp$ PQ, $\therefore$ it is VN. Also it passes through the pt. of contact R.

Ex. 3. Let the $\odot$ x be $\perp$ to the $\odot$s a, b, c. Let PP' be a diameter of x. Then the polar of P w. r. to a passes through P'; so for b, c.

Page 72. Ex. 1. In the figure of p. 72, let PQ' and $P'Q$ cut at R. Then, since the polar of P passes through R, the polar of R passes through P; and so through Q.

Ex. 2. The polar of B (being $C'A'$) passes through A' and the polar of C passes through A'. Hence the polar of A' passes through B and C; so for B' and C'.

Ex. 3. The chords of contact, being polars of pts. on the |, pass through the pole of the |.

Ex. 4. Let the | $ABCD$ be called l, and its pole, L. Then the polar LA' of A passes through L and is $\perp$ OA. Hence the pencil $L(A'B'C'D')$ of polars is superposable to the h^c pencil $O(ABCD)$ and is $\therefore$ h^c.

Ex. 5. Let AP cut QR at U. Then the polar of S passes through P; and also through A, since the polar of A passes through S. Hence AP is the polar of S. Hence (SU, RQ) is h^c, $\therefore$ $A(SU, RQ)$ is h^c, $\therefore$ (SP, BC) is h^c.

Ex. 6. Since the polar of Q passes through M, the polar of M passes through Q and (being $\perp$ the radius OM) is QU, and $\therefore$ passes through U. Hence the polar of U passes through M and is $\therefore$ PM; so the polar of V is PN. Hence the polar of P (on PM and PN) is UV. Hence UV is the tangent at P.

Ex. 7. Let the tangents from P be PR and PR' and let P' be pt. inverse to P. Then $RP'R'$ passes through Q.

$$\therefore\ PQ^2 = P'P^2 + P'Q^2 = t_1^2 - P'R^2 + P'Q^2$$
$$= t_1^2 + (P'Q + P'R)(P'Q - P'R) = t_1^2 + QR \cdot QR' = t_1^2 + t_2^2.$$

Page 74. § 10. Ex. Let the lines p and q (whose poles are P and Q) meet at R. Then, since the polar of P passes through R, the polar of R passes through P; and so through Q. Hence the pole of PQ (being R) lies on p and q; i.e. PQ is conjugate to p and q.

Page 74. § 11. Ex. Let the $\perp$ diameters be AOA' and BOB'. Let the pts. at infinity on AA', BB' be I, J. Then (AA', OI) is h^c. Hence the polar of I, passing through O and being $\perp OA$, is OJ; so the polar of J is OI. Hence OIJ is a self-conjugate Δ.

Page 75. § 12. Ex. 1. Let the polar $\odot$ cut AC in P, P'. Then, since AB is the polar of C, (AC, PP') is h^c; so for BC, AB.

Ex. 2. Since BC is the polar of A, A and X are conjugate pts. Now see p. 72, § 9, end.

Ex. 3. Let $A = 90°$. Then $\rho^2 = HA \cdot HD, = 0$ in this case, for H coincides with A.

Page 75. § 13. Ex. $HO^2 = R^2 + (\rho\sqrt{2})^2$. With H as centre and $\rho\sqrt{2}\ (= r)$ as radius, describe a $\odot$, c. Then, since $HO^2 = R^2 + r^2$, the $\odot$s PQR and c are $\perp$. Also H and ρ (and $\therefore c$) are given.

Page 76. Ex. With the figure of p. 75, let the chord PWQ cut UV at I (at infinity). Then, since UV is the polar of W, (PQ, WI) is h^c, $\therefore W$ bisects PQ.

End of Chapter VI

Ex. 1. Take D such that (AC, BD) is h^c. Then, since $P(AC, BD)$ is h^c, if PB bisects $\angle APC$, PD also bisects it. Hence $BPD = 90°$. Hence the locus of P is the $\odot$ on BD as diameter. Also if P is any pt. on this $\odot$, $\angle BPD = 90°$ and $P(AC, BD)$ is h^c; hence PB bisects $\angle APC$.

Ex. 2. The $\odot$ on PQ as diameter is $\perp$ the given $\odot$. Hence the tangent from R to the $\odot$ is equal to RP.

Ex. 3. Since $OU : OP :: OP : OV$, $\therefore \angle OPU = OVP$, and $OPU = OQP$, $\therefore OVP = OQP$.

Ex. 4. (PP', QQ') is h^c because its orthogonal proj^n (AA', BB') is h^c; hence $O(PP', QQ')$ is h^c. Also if T is the pt. of contact of PP', $\angle AOP = POT$ and $TOP' = P'OA'$. Hence $POP' = \frac{1}{2}(AOT + TOA') = 90°$. Hence OP, OP' bisect QOQ'. Hence $OQ : OQ' :: QP' : P'Q' :: BA' : A'B'$ is const.

Ex. 5. Now $PA : PC :: AB : CD$, which is given. Hence P lies on a $\odot$; so '$PB : PD =$ const.' gives another $\odot$ on which P lies. Hence there are two positions of P. For if X is one of the int^{ns} of the above $\odot^s$, then $XA : XC :: XB : XD :: AB : CD$; hence XAB and XCD are similar.

Ex. 6. Let BN and CM cut at R. Then, by p. 68, Ex. 3, $AMR = 90° = ANR$. Hence $AMNR$ is cyclic.

Ex. 7. Let the int^n be A and the diameters BB', CC'. Let AC, AC' cut the $\odot$ on BB' again at D, D'. Then $DAD' = 90°$, since $CAC' = 90°$. Hence DD' is a diameter. Let O bisect BB'; then $\angle ODA = OAD = AC'C$ since OA touches the $\odot$ on CC'. Hence the Δ^s $D'OC'$ and $D'AD$ are similar. Hence $\angle D'OC' = D'AD = 90°$. Hence AC passes through D, an end of the $\perp$ diameter DD'.

Ex. 8. Since $PQ : PT :: PT : PR$, the Δ^s PQT and PTR are similar, $\therefore$ $QT : RT :: PT : PR$. So $QT' : RT' :: PT' : PR$. Also $PT = PT'$.

Ex. 9. The poles of $|^s$ through a given pt. lie, of course, on the polar of the pt.

Ex. 10. Let the tangents be TL, TM. Then, since the polar of T passes through S, the polar of S passes through T. It also passes through P; and hence is TP. Let TP cut LM at N. Then (SN, ML) is h^c, $\therefore$ $T(SN, ML)$ is h^c, $\therefore$ (SP, RQ) is h^c.

Ex. 11. Take the centre O of the $\odot$. Let PQ cut RS, OS at M, N. The polar of N (which lies on PQ the polar of R) passes through R and is $\perp$ ON; and hence is RS. Hence (NM, PQ) is h^c. Also, since $S(NM, PQ)$ is h^c and $NSM = 90°$, SR bisects PSQ.

Ex. 12. Let EF be the third diagonal. Then E, F, being h^c pts. of the inscribed quadrangle, are conjugate. Hence the $\odot$ on EF as diameter is $\perp$ given $\odot$. Now see p. 73, Ex. 7.

Ex. 13. For UVW (see p. 76) is self-conjugate w. r. to the $\odot$, i.e. the $\odot$ is the polar $\odot$ of UVW. Hence the centre of the $\odot$ is the orthocentre of UVW.

Ex. 14. The $\odot$ on PQ as diameter is $\perp$ given $\odot$ (centre O, radius r), $\therefore$ if C bisects PQ, $OC^2 = CP^2 + r^2$. Hence

$$t^2 = OU^2 - r^2 = OU^2 + CP^2 - OC^2 = CP^2 - CU^2$$
$$= (CP + CU)(CP - CU) = QU \,.\, UP.$$

Ex. 15. The polar of B w. r. to the polar $\odot$ is CA; hence B, C are conjugate pts. Hence the $\odot$ on BC as diameter is $\perp$ polar $\odot$. So for the rest.

Ex. 16. Let the centre and radii of the $\odot^s$ be A, B, C, D and a, b, c, d. Then we are given that $AB^2 = a^2 + b^2$, $AC^2 = a^2 + c^2$, $BD^2 = b^2 + d^2$, $CD^2 = c^2 + d^2$, $\therefore$ $AB^2 - AC^2 = BD^2 - CD^2$, $\therefore$ $AD \perp BC$. So for the rest.

CHAPTER VII

Page 80. Ex. 1. The sum of the powers (viz. $PA^2 - a^2 + PB^2 - b^2$) is const. if $PA^2 + PB^2$ is const. Now see p. 36, Ex. 1.

Ex. 2. B, C lie on the circumcircle and E, F on the N.P.C. Also $PB \,.\, PC = PE \,.\, PF$, since B, C, E, F are concyclic. Hence P (and so Q, R) lies on the r. a.

Ex. 3. (i) Let the $\odot^s$ BCA', CAB' meet again at O. Then $\angle AOB = 360° - BOC - AOC = 360° - (180° - A') - (180° - B') = A' + B' = 120° = 180° - C'$. Hence $\odot$ ABC' also passes through O.

(ii) Join O to A, B, C, A', B', C'. Then $\angle AOC + COA' = 120° + CBA' = 180°$, $\therefore$ A, O, A' are collinear; so BB', CC' pass through O. Now consider the Δ^s ACA' and $B'CB$; then $AC = B'C$, $CA' = CB$ and $\angle ACA' = C + 60° = B'CB$. Hence $AA' = BB'$, $= CC'$ similarly.

(iii) Let the centres of the ⊙s BCA', CAB', ABC' be O_1, O_2, O_3. Then $O_1O_2 \perp OC$ and $O_2O_3 \perp OA$, $\therefore \angle O_1O_2O_3 = 180° - COA = 60°$. So for the other angles.

Ex. 4. See the figure on p. 16. $CX = s - c = BX_1$. Hence A' bisects XX_1. Hence the tangents from A' to i and i_1 are equal. Hence A' is on the r. a. of i and i_1. The r. a. is $\perp II_1$, i.e. $\perp$ to a bisector of BAC, i.e. $\perp$ to a bisector of the $\parallel$ angle $B'A'C'$, i.e. coincides with a bisector of $B'A'C'$.

Again, $BX_3 = s - a = CX_2$. Hence $A'X_3 = A'X_2$. Hence the r. a. of i_2, i_3 passes through A'. Also this r. a. is $\perp I_2I_3$, i.e. $\parallel AI_1$, i.e. $\parallel$ to the other bisector of $B'A'C'$.

Page 81. Ex. 1. Call the fixed pts. A, B. Let two such ⊙s, one fixed and the other variable, cut the fixed ⊙ in P', Q' and P, Q. Then the r. a.s $P'Q'$, PQ, AB concur; i.e. PQ passes through the fixed intn of $P'Q'$ and AB.

Ex. 2. Let the pts. A_1, A_2 on BC, and B_1, B_2 on CA and C_1, C_2 on AB be such that $A_1A_2B_1B_2$, $B_1B_2C_1C_2$, $C_1C_2A_1A_2$ lie on the circles c, a, b. Then the r. a. of c and a is B_1B_2; and so on. Hence CA, AB, BC concur if a, b, c are all different. Suppose, then, that a and b coincide in d. Then $BB_1C_1C_2A_1A_1$ lie on d.

Ex. 3. Let the ⊙s be a, b, c, d. Consider a, b, c; then the r. a.s $(ab)(bc)(ca)$ concur, in D, say. Proceeding thus, we see that the quadrangle is $ABCD$, A being the intn of (bc, cd, db), and B of (cd), (da), (ac), and C of (ab), (bd), (da).

Ex. 4. The ⊙ on BC as diameter passes through F; hence the power of H w. r. to it is $HC \cdot HF$. So for the rest. Also $HC \cdot HF = HA \cdot HD = HB \cdot HE$.

Ex. 5. Let the polars of P w. r. to the ⊙s a, b, c meet at P'. Then P, P' are conjugate pts. w. r. to each ⊙; hence the ⊙ on PP' as diameter is $\perp a, b, c$. Hence P lies on this $\perp$ ⊙, i.e. on the radical ⊙.

Page 83. Ex. The difference, viz. $(PA^2 - a^2 - PB^2 + b^2)$, is const. if $PA^2 - PB^2$ is const. Now see p. 84, § 5.

Page 85. Ex. 1. This is practically the same as p. 81, Ex. 1.

Ex. 2. $a^2.BC+b^2.CA+c^2.AB+BC.CA.AB$
$= (OA^2-k).BC+(OB^2-k).CA+(OC^2-k).AB$
$+BC.CA.AB$ [since $OA^2-a^2 = OB^2-b^2 = \ldots = k$]
$= OA^2.BC+OB^2.CA.OC^2.AB+BC.CA.AB$
$-k(BC+CA+AB) = 0$ by p. 83, § 4.

Ex. 3. $OA^2 = OL^2+a^2 > OL^2$, $\therefore$ $OA > OL$.

Ex. 4. Bisect one such tangent LT at P. Then $PL = PT$. Hence P has the same power w. r. to the pt.-⊙ L and the other ⊙. Hence the locus of P is the r. a. of the system.

Ex. 5. U and U' are conjugate w. r. to the pt.-⊙, L. Hence LU' is the polar of U, $\therefore$ $\angle ULU' = 90°$.

Ex. 6. Let the ⊥ˢ be PM, QN. Then $PL^2 = 2.PM.AL$ and $QL^2 = 2.QN.AL$, $\therefore$ $PL^2.QL^2 = 4.PM.QN.AL^2$. Also $PL.LQ$ and AL are known.

Ex. 7. Let PP' cut the r. a. at X. Then $XL^2 = XP.XP' = XQ.XQ'$. Hence XL touches the ⊙ PLP' (and the ⊙ QLQ') at L.

If the ⊙ˢ are on the same side of the r. a., $\angle PLQ = XLQ - XLP = XQ'L - XP'L = P'LQ'$.

If on opposite sides (taking the pts. in the order $P'PQQ'$), $\angle PLQ = PLX + XLQ = XP'L + XQ'L = 180° - P'LQ'$.

Ex. 8. Let PQ cut the r. a. at X. Then $XP = XQ = XL$, since X has the same power for each ⊙. Hence PLQ is a semi-⊙. So for L'.

Page 86. Ex. 1. For

$$OL^2 = OX.OY = OX'.OY' = \ldots = OL'^2.$$

Ex. 2. The polars of L w. r. to c_1 and c_2 coincide. Hence PS is the polar. Let PS cut the | of centres ⊥ˡʸ at N, and cut c_2 again at S'. Then

$$LS^2 - LP^2 = NS^2 - NP^2 = (NS+NP)(NS-NP) = S'P.PS = QP.PR.$$

Page 87. Ex. This is the same as p. 81, § 2.

Page 88. Ex. 1. For the centre of the ⊥ ⊙ has the same power w. r. to each ⊙, viz. the square of its radius.

Ex. 2. Let the r. a.^s meet at P. From P draw the tangents PT, PT' to any ⊙^s of the two systems and the tangent PA to the common ⊙. Then $PT = PA = PT'$. Hence the circle with centre at P and radius PA is ⊥ all the ⊙^s.

Page 90. Ex. 1. In § 11 it is proved that the ⊙ on CC' as diameter is ⊥ to the polar ⊙^s of ABC and $A'B'C$; similarly it is ⊥ to the polar ⊙^s of $B'C'A$, $C'A'B$. Also, by p. 69 Ex., it is ⊥ to the ⊙ circumscribing the h^c Δ. So the ⊙ on AA' as diameter is ⊥ to these five ⊙^s. Hence these five ⊙^s are coaxal. Hence their centres are collinear, viz. the orthocentres and the circumcentre of the h^c Δ.

Ex. 2. Project LMN to infinity. Then, since (AA', LL') is h^c, L' bisects AA' in the new figure; so M' bisects BB' and N' bisects CC'. Hence L', M', N' are collinear in the new figure and ∴ in the old.

Page 92. § 13. Ex. 1. Construct two ⊙^s b and b' of the system which is ⊥ to the given system. Let c be the radical ⊙ of b, b' and the given ⊙, a. Then c, being ⊥ b and b', belongs to the original system and is also ⊥ a.

Ex. 2. Let a be the given ⊙. Take any two coaxals, c_1, c_2, of the system, and let the radical ⊙ r of (a, c_1, c_2) cut a at P and Q; then r, being ⊥ c_1 and c_2, is ⊥ to each of the coaxals. Draw a coaxal x through P. Then x, being ⊥ r which is ⊥ a, touches a. So Q gives another solution.

Page 92. § 14. Ex. 1. Through P draw a ⊙ z (with centre Z) such that the given | is the r. a. of the given ⊙ and z. Then, if X is the centre of the given ⊙, $PQ^2 = 2PN.XZ$, $\therefore XZ = \frac{1}{2}PQ^2/PN$ which is known. Hence Z is known. Hence the locus of P is the known ⊙ z.

Ex. 2. Let the centres of the ⊙^s be X_1, X_2, X_3, X_4. Then $t_1{}^2 = 2PN.X_1Z$, $t_2{}^2 = 2PN.X_2Z$, $t_3{}^2 = 2PN.X_3Z$, $t_4{}^2 = 2PN.X_4Z$. Hence $t_1{}^2t_2{}^2 = t_3{}^2t_4{}^2$ gives $X_1Z.X_2Z = X_3Z.X_4Z$ as the equation to determine Z. Now see p. 59, Ex. 1 (Z being O).

Ex. 3. Let the ⊙s ABC (with centre X) and PQR (with centre Y) touch at T. Draw $AN \perp$ the tangent at T (i.e. to the r. a.). Then $AP^2 = 2AN.XY$. Again T is a limiting pt. Hence $AT^2 = 2AN.XT$. Hence $AP^2/AT^2 = XY/XT$, $= BQ^2/BT^2 = CR^2/CT^2$ similarly. Hence $AP/AT = BQ/BT = CR/CT$ $\therefore$ $AP.BC \pm BQ.CA \pm CR.AB = 0$, if $AT.BC \pm BT.CA \pm CT.AB = 0$; which is true by Ptolemy's theorem.

Page 96. Ex. 1. Let the | of centres cut b again at A, and let a and b intersect at B and D. Then, if we proceed from A to B, then from B to A, then from A to D, and finally from D to A, we get one such crossed quad., $ABADA$. Now move A continuously along b and, in any position, draw the tangents from A to a, cutting b again at B and D. Then, by Poncelet's theorem, the other tangents to a from B and D will meet at A' (on b), so that $ABA'D$ is a crossed quad.

Ex. 2. Since L, M, N, R are the points of contact of the tangents AB, BC, CD, DA of the inner ⊙, by p. 76, § 16 the four internal diagonals of $LMNR$ and $ABCD$ concur (at O, say) and the two external diagonals coincide (in l, say). Since $LMNR$ is inscribed in the inner ⊙, l is the polar of O w. r. to the inner ⊙; and similarly w. r. to the outer ⊙, since $ABCD$ is inscribed in the outer ⊙. Hence O has the same polar w. r. to both ⊙s and is $\therefore$ a limiting pt. of the ⊙s. Also it is the limiting pt. inside both ⊙s, since AC, BD intersect inside the ⊙s. Hence it is a fixed pt. Now the pt.-⊙, O, and the inner ⊙ are coaxal with the outer ⊙. Hence, by p. 92, § 14, $DR:DO::AR:AO$ (for the tangent from D to the pt.-⊙, O, is DO; so for AO). Hence RO bisects AOD; so LO bisects AOB. Hence $\angle LOR = \frac{1}{2}(DOA + AOB) = 90°$.

Page 97. Ex. 1. Let the ⊙s APP', AQQ' cut again at B and let the ⊙ ABR cut AP' at X'. Then $PQ:QR::P'Q':Q'X'$; but $PQ:QR::P'Q':Q'R'$. Hence X' coincides with R'; and so on.

Ex. 2. Consider the pedal | of B w. r. to the Δ APP'; it joins the proj[ns] of B on AP and AP'. Hence it is also the pedal | of B w. r. to AQQ', ARR', ... Also the proj[ns] of B on PP', QQ', RR', ... lie on this pedal |.

End of Chapter VII

Ex. 1. Let the centres be A, B, C and the radii a, b, c. Then, since C is on the r. a. of 1 and 2, $CA^2 - a^2 = CB^2 - b^2$; so $AB^2 - b^2 = AC^2 - c^2$,

$\therefore$ $BC^2 - BA^2 = CA^2 + b^2 - a^2 - AC^2 + c^2 - b^2 = c^2 - a^2$.

Hence the r. a. of 3 and 1 passes through B.

Ex. 2. Let the $\odot$ x, with centre X and radius x, cut the circle a with centre A and radius a at P, Q so that PQ is a diameter of x. Then $AX \perp PQ$, since X bisects PQ; hence $XA^2 = AP^2 - PX^2 = a^2 - x^2$. So for the $\odot$s b, c. Hence $XA^2 - a^2 = XB^2 - b^2 = XC^2 - c^2 = -x^2$. Hence X has the same power w. r. to a, b, c, i. e. X is the radical centre. Also this power $(= -x^2)$ is $-$. Hence the r. c. must be an internal pt. Conversely with the r. c. R as centre and radius $r = \sqrt{a^2 - RA^2}$, describe a $\odot$ r, cutting a at P, Q. Then $AP^2 = a^2 = RA^2 + RP^2$, $\therefore$ $\angle ARP = 90°$; so $ARQ = 90°$. Hence RPQ is a |, i. e. PQ is a diameter of r. But $r^2 = a^2 - RA^2 = b^2 - RB^2$, since R is the r. c. Hence $r = \sqrt{b^2 - RB^2}$. Hence r cuts b also at the ends of a diameter; so for c.

Ex. 3. Let $TP(=p)$ be the tangent from T to the $\odot$ a. Then $\quad p^2 = TP^2 = TA^2 - a^2 = TA^2 - OA^2 + k,$ where O is the int[n] of the r. a. and line of centres and

$$k = OA^2 - a^2 = OB^2 - b^2 = \dots.$$

Hence
$$\begin{aligned} & p^2 . BC + q^2 . CA + r^2 . AB \\ &= (TA^2 - OA^2 + k) BC + \dots \\ &= TA^2 . BC + TB^2 . CA + TC^2 . AB - OA^2 . BC \\ &\qquad - OB^2 . CA - OC^2 . AB \\ &\quad + k(BC + CA + AB) = -BC . CA . AB + BC . CA . AB \end{aligned}$$

(by p. 33) $= 0$.

Ex. 4. If P, P' are conjugate w. r. to the $\odot$ a, then the $\odot$ (r with centre R) on PP' as diameter is $\perp$ a. Hence the pt. R has the power r^2 w. r. to a; and so w. r. to $b, c, \ldots$. Hence R has the same power w. r. to each of the $\odot$s; i.e. the r. a.s concur at R.

Ex. 5. Remembering that the pt.-$\odot$s L, M are $\odot$s of the system, and calling the centre of the given $\odot$ C,

$$\frac{PA^2-PL^2}{PA^2-PM^2}=\frac{2PN.CL}{2PN.CM}=\frac{CL}{CM}=\frac{2AN'.CL}{2AN'.CM}=\frac{AL^2}{AM^2},$$

where PN and AN' are $\perp$s on the r. a.

Ex. 6. The $\odot$ is $\perp$ to the $\odot$s a and b on AA' and BB' as diameters. Hence its centre lies on the r. a. of a and b.

Ex. 7. Let the centre of the required $\odot$ x be X; then X lies on the r. a. of the $\odot$s a, b. Let XP be the radius of x which touches a; then $x^2 = XP^2 = XA^2 - a^2 = XO^2 + OA^2 - a^2$. But $OA^2 - a^2$ is known and x is given; hence OX and $\therefore$ X is known.

Ex. 8. This is proved in the solution of p. 96, Ex. 2.

Ex. 9. Let the $\odot$s APP', AQQ' meet again at B; and let the $\odot$ ABR cut PP' at X'. Then (PP', QQ', RX') is an involution; and so is (PP', QQ', RR'). Hence X' and R' coincide, since each corresponds to R in the involution determined by PP', QQ'.

Ex. 10. Let the $|$ cut the $\odot$s at P, Q and R, S. Let the tangent at P cut those at R, S at X, Y; let the tangent at Q cut those at R, S at Z, U. Then, by p. 92, § 14, X, Y, Z, U lie on a $\odot$ coaxal with the given $\odot$s if $XP : XR :: YP : YS :: ZQ : ZR :: UQ : US$. But if the acute angles at P and Q are equal to a and those at R and S to β, $XP : XR = \sin XRP : \sin XPR = \sin\beta : \sin a$; and so for the rest.

Ex. 11. By p. 97, Ex. 2, the proj^{ns} of B on $PP', QQ', RR', \ldots$ are collinear. Here P and P' coincide and so on. Hence the proj^{ns} of B on the tangents at P, Q, R are collinear. Now see p. 25, § 14.

Ex. 12. Let the tangents be TP and TQ. Then $\angle TPQ = ABP = ABQ = TQP$, $\therefore TP = TQ$. Hence the locus of T is the r. a.

Ex. 13. As in the solution of p. 90, Ex. 1, we prove that each of the ⊙s on AA', BB', CC' as diameter is ⊥ to each of the polar ⊙s of ABC, $A'B'C$, $B'C'A$, $C'A'B$. Hence the two systems of ⊙s are ⊥ coaxal systems. Hence the centres of the first system (viz. the centres of AA', BB', CC') lie on a | ⊥ to the | on which lie the centres of the second system (viz. the orthocentres).

Ex. 14. The given ⊙ is ⊥ to the system of coaxal ⊙s through A, B. Hence its centre X is on AB; and if x is its radius $x^2 = XA \cdot XB$. Hence A, B are inverse w. r. to it.

Ex. 15. Let A, B be the given pts. and C the given pt. On the given ⊙ take any pt. P and let the ⊙ ABP cut the given ⊙ again at Q. Let PQ, AB meet at R. Let RC cut the given ⊙ at X, Y. Then $RX \cdot RY = RP \cdot RQ = RA \cdot RB$. Hence A, B, X, Y are concyclic; i.e. ABX is the required ⊙.

CHAPTER VIII

Page 100. Ex. 1. For II_1 is the line of centres; and BC is a transverse common tangent.

Ex. 2. Let an isogonal | cut one ⊙ at P, Q and the other ⊙ at P', Q'. Let the tangents at P, P' be PT, $P'T'$; then by hyp. $\angle TPQ = T'P'Q'$. First let the centres, A, B', of the ⊙s be on the same side of PP'. Let PP' cut the | of centres at O, P being the pt. nearest O. Then $\angle OPA = 90° + TPQ = 90° + T'P'Q' = OP'B'$. Hence the Δs OPA and $OP'B'$ are similar. Hence $OA : OB' :: AP : B'P'$. Hence O is the external c. of s.

Now let A, B' be on opposite sides of PP'. Then, as before, $OA : OB' :: AP : B'P'$, but O is between A and B'. Hence O is the internal c. of s.

Ex. 3. Let PP' cut the line of centres (A, B') in O. Then, since $AP \parallel B'P'$, $OA : OB' :: AP : B'P'$. Hence PP' passes through the external c. of s., S; so QQ' passes through S. Similarly PQ' and $P'Q$ pass through S'.

Ex. 4. In the figure of p. 99, draw $A'N \perp AP$. Then

$$\begin{aligned} t_1^2 &= P'P^2 = A'N^2 = A'A^2 - AN^2 = d^2 - (a-b)^2 \\ &= (d+a-b)(d-a+b) \\ &= (AA' + BA - B'A')(AA' - AC + A'C') = BB' . CC' \end{aligned}$$

where $d = AA'$, $a = AP$, $b = A'P'$.

So if QQ' is a transverse common tangent, drawing $A'N \perp AQ$,

$$\begin{aligned} t_2^2 &= Q'Q^2 = A'N^2 = A'A^2 - AN^2 = d^2 - (a+b)^2 \\ &= (d+a+b)(d-a-b) \\ &= (AA' + BA + A'C')(AA' - AC - B'A') = BC' . CB'. \end{aligned}$$

Page 101. Ex. Drawing the figure of § 4, the diagonal SS' is divided h^{ly} by L, L'.

Page 102. Ex. 1. The $\odot^s$ are homothetic w. r. to S. Also the | through S corresponds to itself. Hence the poles of the | are corresponding homothetic pts.; and hence are collinear with S.

Ex. 2. $PP' = SP' - SP = k . SP - SP = (k-1) SP$; so $QQ' = (k-1) SQ$ and $TT' = (k-1) ST$. Also $SP . SQ = ST^2$.

Ex. 3. The locus of B' is a $\odot$. Also $CA = 2 . CB'$. Hence the locus of A is a $\odot$.

Page 103. Ex. 1. H is the external c. of s. of the $\odot$ ABC and the N.P.C. And Q, A' are on these $\odot^s$. Hence $HQ : HA' :: R : n :: 2 : 1$.

Ex. 2. Let PQ be the diameter and p, q the pedal lines of P, Q. Then by p. 26, Ex. 4, p passes through the centre P' of HP and q through the centre Q' of HQ where H is the orthocentre of the given Δ. Also the angle between p and q is the angle subtended by PQ at any pt. on the $\odot$, $\therefore$ $p \perp q$. Hence if p, q meet at R, $\angle P'RQ' = 90°$. Again H is the external c. of s. of the given $\odot$ and the N.P.C. Hence, since $HP' = \frac{1}{2} HP$, $HQ' = \frac{1}{2} HQ$, P' and Q' are the

pts. on the N.P.C. corresponding to P and Q on the given ⊙. Hence $P'Q'$ is a diameter of the N.P.C. Hence R lies on the N.P.C., since $\angle P'RQ' = 90°$.

Page 104. **Ex. 1.** In the figure on p. 103, draw the tangents OT, OT' to the ⊙s from the pt. O on the ⊙ of s. Then $OA : OA' = a : a' = AT : A'T'$. Hence the right-angled Δs OAT and $OA'T'$ are similar. Hence $OT : OT'$ $:: AT : A'T' :: a : a'$ and $2\angle TOA = 2\angle T'OA'$.

Ex. 2. Let the given ⊙s meet at P and Q. Then PA $: PA' = a : a'$. Hence P is on the ⊙ of s. So Q. Hence the three ⊙s pass through P and Q.

Ex. 3. Let the centres of the three ⊙s be A, B, C and the radii a, b, c. Let the ⊙s of s. of a, b and b, c meet at P and Q. Then $PA : PB = a : b$ and $PB : PC = b : c$, $\therefore$ PA $: PC = a : c$. Hence P (and so Q) lies on the ⊙ of s. of c, a.

End of Chapter VIII

Ex. 1. Since $PQ : QA : AP :: P'Q' : Q'A' : A'P'$, the Δs PQA, $P'Q'A'$ are similar. Hence $\angle APQ = AQP = A'P'Q'$ $= A'Q'P'$. Hence (since $PQ \parallel P'Q'$) AP is $\parallel$ to $A'P'$ or $A'Q'$; say $AP \parallel A'P'$, then $AQ \parallel A'Q'$. Now see p. 100, Ex. 3.

Ex. 2. Let PQ pass through S; and let AP, BQ meet at W (at infinity). Let PB, QA meet at X. Consider the quadrilateral $XAWBX$. The diagonal AB is cut h^{ly} by PQ and XW. But (AB, SS') is h^c. Hence XW passes through S'; and it is $\parallel$ PA, i.e. $\perp$ PQ. Hence $S'X \perp PQ$. Let it cut PQ at N. Then (NS', XW) is h^c, $\therefore$ X bisects NS'.

Ex. 3. $t_1^2 = d^2 - (a-b)^2$, $t_2^2 = d^2 - (a+b)^2$, [p. 100, Ex. 4]

$$\therefore \; t_1^2 - t_2^2 = d^2 - a^2 + 2ab - b^2 - d^2 + a^2 + 2ab + b^2$$
$$= 4ab = d_1 d_2.$$

Ex. 4. Let the common tangents touch one ⊙ at A, B, C, D; and, comparing with the figure on p. 76, arrange that Q of p. 76 is S and Q' is S'. Then BA, CD, QQ', RR' meet at U which is L' as in § 4 of p. 101. So AC, BD, PP', QQ'

meet at W which is L. Lastly AD, BC, PP', RR' meet at V which is at infinity in a direction $\perp$ SS'.

Ex. 5. Let AP cut BC at P'. With A as homothetic centre and P, P' as corresponding pts., form the $\odot$, c' homothetic to the inscribed $\odot$, c. Then the tangents at P and P' are $\parallel$, $\therefore$ c' touches BC. Also AB, AC, being the tangents to c from the homothetic centre, touch c'. Hence c' is the $\odot$ escribed to BC.

Ex. 6. Let l, m be any $|^s$ through S. Then the corresponding $|^s$ l', m' coincide with l, m. Now if l, m are conjugate w. r. to c, then l', m' are, by similarity, conjugate w. r. to c', i.e. l, m are conjugate w. r. to c'.

Ex. 7. Since H is the external c. of s. of the circumcircle and the N.P.C., it follows that if any $|$ through H cut these $\odot^s$ at Q, Q', then $HQ = 2 . HQ'$, $\therefore$ Q' bisects HQ.

Ex. 8. We know that A, B, C are the feet of the $\perp^s$ from I_1, I_2, I_3 on I_2I_3, I_3I_1, I_1I_2. Hence I is the orthocentre of $I_1I_2I_3$. Hence I is a c. of s. of the $\odot^s$ $I_1I_2I_3$ and ABC.

Ex. 9. Let S and S' be the c^s. of s. of the circles a, b. Then the $\odot$ of s. of a, b is the $\odot$ on SS' as diameter which is $\perp$ to any $\odot$ through A, B since (SS', AB) is h^c and $\therefore$ $\perp$ to the $\odot$ ABC. So for the rest.

Ex. 10. With the figure of p. 24, X and A' on the N.P.C. correspond to A and Q on the circumcircle. Hence, since $A'X$ is a diameter of the one, AQ is a diameter of the other, and $\therefore$ passes through O.

Ex. 11. Let H_1, H_2 be the orthocentres of the Δ^s A_1BC, A_2BC. Then $A_1H_1 = 2OA' = A_2H_2$. Hence $A_1H_1 =$ and $\parallel$ A_2H_2. Hence $A_1H_1H_2A_2$ is a $\parallel^m$. Hence A_1H_2 and A_2H_1 are bisected at the same pt.; and through this pt. pass the two pedal $|^s$ by p. 26, Ex. 4.

Also (as in Ex. 7) the N.P.C. of A_1BC bisects H_1A_2; so for A_2BC.

Ex. 12. As in Ex. 2 of p. 18, $II_1 = 2IL$. Hence the locus of I_1 is a $\odot$ homothetic with the circum$\odot$, taking I as h. c. and 2 as h. r.; so for I_2 and I_3.

Ex. 13. Since BC is given and the angle A, the locus of A is a ⊙ through B, C. Also $A'G = \frac{1}{3}A'A$. Hence the locus of G is also a ⊙.

CHAPTER IX

Page 107. Ex. $OP_1 = k_1/OP$, $OP_2 = k_2/OP_1$, $OP_3 = k_3/OP_2$, and so on, P, P_1, P_2, P_3, ... being collinear. Hence $OP_2 = k \,.\, OP$ where $k = k_2/k_1$, and $OP_3 = k'/OP$ where $k' = k_3k_1/k_2$. Hence P_2 generates a figure homothetic, and P_3 a figure inverse, to that generated by P. And so on.

Page 109. § 2. Ex. $OP \,.\, OA = OQ \,.\, OB$ and $OA \,.\, OA' = OB \,.\, OB'$, $\therefore$ $OP : OQ :: OB : OA :: OA' : OB'$. Hence $PQ \parallel A'B$, and so on.

Page 109. § 3. Ex. 1. The Δ^s OPQ and $OQ'P'$ are similar. Hence $p : p' :: PQ : Q'P'$.

Ex. 2. Suppose ABC to be inverted into the triangle $A'B'C'$ of given form; then the ratios $A'B' : B'C'$ and $B'C' : C'A'$ are given. But $A'B' : B'C' :: k \,.\, AB/OA \,.\, OB \div k.BC/OB.OC :: AB.OC : BC.OA$. Hence the ratio $OC : OA$ is known, since AB and BC are given. Hence (p. 66) the locus of O is a known ⊙. So from $B'C' : C'A'$ we get another ⊙ on which O lies. Conversely, take O at either of the int^{ns} O_1, O_2 of these ⊙^{s}. Then the ratios $A'B' : B'C'$ and $B'C' : C'A'$ have the required values; hence $A'B'C'$ has the required form.

Ex. 3. We have to show that the int^{ns} O_1 and O_2 are inverse w. r. to the ⊙ ABC. Let O_3 be the inverse of O_1 w. r. to the ⊙ ABC. Then (p. 65) $AO_1 : AO_3 :: BO_1 : BO_3$, i. e. $O_3A : O_3B :: O_1A : O_1B$. So $O_3C : O_3A :: O_1C : O_1A$. Hence O_3 is the other int^n of the ⊙^{s}; i. e. O_2 is O_3.

Page 110. Ex. 1. Inverting w. r. to A, the given ⊙^{s} b, c invert into the $|^s$ b', c'; the $|^s$ l, m become the ⊙^{s} l', m' through A. Hence in the new figure l' touches b', c' at P', Q', and m' touches b', c' at R', S'. Hence $P'Q' \parallel R'S'$ by

symmetry. Hence in the given figure the ⊙s APQ, ARS touch at A.

Ex. 2. Inverting w. r. to A, we have a | through B' ⊥ the | $C'D'$, a | through C' ⊥ the | $B'D'$ and a | through D' ⊥ the | $B'C'$. Hence these |s concur, at H', say. Then in the given figure the three ⊙s concur at A and H.

Ex. 3. Invert w. r. to P. Then the ⊙ ABC inverts into a |; hence A', B', C' are collinear. But $PL : BC = p : B'C'$ where p is the ⊥ from P to the | $A'B'C'$; and so on. Hence we have to prove that $B'C' + C'A' + A'B' = 0$.

Page 112. Ex. 1. Through O draw the tangent OT to c. Then OT touches c' at the pt. T', inverse to T. Hence if C' is the inverse of C, then $OC \,.\, OC' = OT \,.\, OT'$. Hence $CC'T'T$ is cyclic, $\therefore$ $\angle OC'T' = OTC = 90°$. Hence $C'T'$ is the polar of O w. r. to c'.

Ex. 2. Invert w. r. to O. Then the ⊙s invert into ⊙s and the intns into the intns. Now, in the new figure the r. a.s AA', BB', CC' concur. Hence in the given figure the ⊙s OAA', OBB', OCC' have a second common pt.

Ex. 3. Invert w. r. to any pt., O, on the radical ⊙, r. r inverts into a | ⊥ to each of the new ⊙s. Hence the new ⊙s have a common diameter, i.e. have their centres collinear.

Ex. 4. The polar, p, of O is a | through T (see Ex. 1) ⊥ the line of symmetry, CC'. Hence its inverse, p', is a ⊙ through O and T' with centre on CC'.

Ex. 5. Let c be a ⊙r section of the cone. Then c', the inverse of c, is a ⊙. Also c' lies on the cone; for the inverse of the pt. P on c is a pt. P' on the cone and on c'. Hence c' is a ⊙r section of the cone. Also ∥ sections of a cone are similar. Hence we get two systems of ⊙r sections of the cone. These only coincide when the cone is right ⊙r.

Ex. 6. Invert w. r. to an intn O of the given ⊙s a, b. Then the variable ⊙s x, y touching at P invert into the ⊙s x', y' touching at P' and touching the |s a', b'. Then if the

$\odot^s$ lie in the same angle, P' lies on the bisector of this angle; but if in different angles, P' lies on a' or b'. Hence the complete locus is a', b' and their bisectors. Hence in the given figure the locus is a, b and the $\odot^s$ through O which bisect the angle at O between the $\odot^s$ a, b.

Page 113. § 6. Ex. 1. Suppose we invert the $\odot^s$ c_1, c_2 with radii r_1, r_2 into $\odot^s$ with radii r'_1, r'_2. Then $r'_1 = kr_1 / (d_1^2 - r_1^2), r'_2 = kr_2 / (d_2^2 - r_2^2)$. Now r_1, r_2, r'_1, r'_2 are known; hence the ratio $(d_1^2 - r_1^2) : (d_2^2 - r_2^2)$ is known, i.e. the ratio of the powers of O w. r. to the two given $\odot^s$ is known. Hence O lies on a known $\odot$ by p. 92. Conversely taking O anywhere on this $\odot$, determine k from $r'_1 = kr_1 (d_1^2 - r_1^2)$; then by the method of obtaining O, we shall also have $r'_2 = kr_2 / (d_2^2 - r_2^2)$.

Ex. 2. As above $r'_1 = r'_2$ if $r_1 / (d_1^2 - r_1^2) = r_2 / (d_2^2 - r_2^2)$ which gives $(d_1^2 - r_1^2) : (d_2^2 - r_2^2)$. Hence as above O may have any position on a certain $\odot$ and k may have any value.

Ex. 3. As in Ex. 1, $r'_1 = kr_1 / (d_1^2 - r_1^2)$ and $r'_2 = kr_2 / (d_2^2 - r_2^2)$ give a $\odot$ on which O must lie. So $r'_2 = kr_2 / (d_2^2 - r_2^2)$ and $r'_3 = kr_3 / (d_3^2 - r_3^2)$ give another $\odot$ on which O must lie. Now take as O either int^n of these $\odot^s$ and determine k from $r'_2 = kr_2 (d_2^2 - r_2^2)$. Then from the properties of the above $\odot^s$, r'_1 and r'_3 have the required values.

Ex. 4. As in Ex. 2, $r'_1 = r'_2$ if O lies on a certain $\odot$. So $r'_2 = r'_3$ if O lies on another $\odot$. Hence if we take as O either int^n of these $\odot^s$, we have $r'_1 = r'_2 = r'_3$, whatever value be given to k.

Page 113. § 7. Ex. 1. We know that A, D are inverse pts. w. r. to the polar $\odot$; and so on. Hence if we invert w. r. to the polar $\odot$, A, B, C invert into D, E, F. Hence the $\odot$ ABC inverts into the $\odot$ DEF w. r. to the polar $\odot$. Hence the circum$\odot$, the N.P.C. and the polar $\odot$ are coaxal.

Ex. 2. If OL cuts BC at A', then A' bisects BC. Also A', L are inverse w. r. to the $\odot$ ABC; and so on. Hence the

$\odot$s LMN, $A'B'C'$ are inverse w. r. to the $\odot$ ABC; i. e. the $\odot$ LMN is coaxal with the N.P.C. and the circum$\odot$.

Page 114. Ex. 1. Suppose the $\odot$ x is $\perp$ $\odot$ a and touches $\odot$ b. Invert w. r. to a. Then a and x invert into themselves. Also b inverts into b'. But x touches b, $\therefore$ x touches b'.

Ex. 2. Let the $\odot$ x, $\perp$ to the given $\odot$s a and b, cut the r. a. of a, b in O and O'. Draw the equal tangents OT, OT' to a, b; and invert w. r. to the $\odot$ with centre O and radius OT. Then a, b invert into themselves. Also x inverts into a | x' $\perp$ to a and b, i. e. into the common diameter $ABCD$. Hence A' which lies on x and a inverts into A or B which lie on x' and a; and so on. Hence for some order of A, B, C, D, AA', BB', CC', DD' concur at O; and so for O'.

Ex. 3. Invert w. r. to the radical $\odot$ of the three $\odot$s a, b, c. Then the $\odot$ x, touching a, b, c, inverts into the $\odot$ x', touching a', b', c', i. e. touching a, b, c.

Page 115. Ex. Let E be one of the int^{ns} of the $\odot$s a, b. Then the $\odot$s of inversion s, s' have S, S' as centres and SE and $S'E$ as radii. Invert w. r. to s. Then a inverts into b and b into a whilst s inverts into itself. Hence s makes the same angle at E with a and b. In exactly the same way, s' also makes the same angle with a and b. Hence s and s' bisect the angles between a and b and are $\therefore$ $\perp$.

Page 116. § 10. Ex. Let the tangents be TP, TQ to a and TL', TM' to a'. Take P', either inverse pt. of P, on a'. Then the tangents at P and P' meet on the r. a. Hence P' is either L' or M'. And so on.

Page 116. § 11. Ex. 1. Let the pts. of contact of x with a, b be P, Q. Then PQ passes through a c. of s., S, say. From S draw the tangent ST to x. Then the $\odot$ with centre S and radius ST is $\perp$ x. Also this $\odot$ is a $\odot$ of inversion of a, b, since $SP \cdot SQ = ST^2$, and hence is coaxal with a, b.

Ex. 2. Let c and d be touched by a at P, P' and by b at Q, Q'. Then we are given that PP' and QQ' pass through

the same c. of s., S, of c and d. Let PQ, $P'Q'$ meet at U. Then c and d touch a, b in the same manner if U is a c. of s. of a and b. Invert w. r. to U, taking P and Q as inverse pts.; then, since $PP'QQ'$ is cyclic, $UP \cdot UQ = UP' \cdot UQ'$, $\therefore$ P', Q' are also inverse pts. Also c and d invert into themselves; for $k = UP \cdot UQ = UP' \cdot UQ'$. Again a inverts into a $\odot$ touching c at the pt. inverse to P, i.e. at Q, and touching d, similarly, at Q'; i.e. a inverts into b. Hence U is a c. of s. of a, b. Also S is on the r. a. of a and b since $SP \cdot SP' = SQ \cdot SQ'$. So U is on the r. a. of c and d.

Page 117. Ex. 1. Let the limiting pts. be L, M. On LM take any pt. O and invert w. r. to O taking $k = OL \cdot OM$; then L, M invert into one another. Also L, M invert into the limiting pts. Hence the limiting pts. L, M invert into the limiting pts. M, L. Hence the coaxal system inverts into a coaxal system having the same limiting pts., i. e. into itself.

Ex. 2. Let the $\odot^s$ x, y through A touch the $\odot$ b and cut at a given angle and meet again at P. Invert w. r. to A. Then P' is the int^n of tangents x', y' to a given $\odot$ b' which meet at a given angle. Hence the locus of P' is a $\odot$, c', concentric with b'. Hence the locus of P is a $\odot$, c, such that A is a limiting pt. of b and c.

Page 118. Ex. 1. Let the variable $\odot$ x touch the $\odot^s$ a, b in the same way and cut the coaxal $\odot$ c. Invert w. r. to a limiting pt., L, of a, b, c. Then the $\odot^s$ a', b', c' are concentric. Also the various positions of x' can be obtained by rotating x' about the common centre. Hence x' in all its positions cuts c' at the same angle. Hence x in all its positions cuts c at the same angle. Also x' is everywhere $\perp$ to a concentric $\odot$. Hence x is everywhere $\perp$ to a coaxal $\odot$.

Ex. 2. From X, the centre of the $\odot$ x, draw $XN \perp$ the r. a.; and let Y be an int^n of x and the r. a. Then since the r. a. is the limit of a coaxal, x cuts the r. a. at a const. angle, ϕ, say; hence $XYN = 90° - \phi$. But XN/XY

$$= \sin XYN = \cos\phi;\ \text{hence } XY \propto XN.$$

Page 119. Ex. 1. Let the ⊙ c pass through the pts. P, Q which are inverse pts. on a, b. Invert with respect to any pt. on c. Then c becomes a | c', and P', Q' become inverse pts. on the ⊙s a', b'. Hence $P'Q'$ passes through a c. of s. of a', b', and hence cuts a', b' at the same angle. Hence c cuts a, b at the same angle. Conversely, if c cuts a, b at the same angle, the line $P'Q'$ cuts a', b' at the same angle. Hence $P'Q'$ passes through a c. of s. of a', b' by p. 100, Ex. 2. Hence, choosing Q' properly, P', Q' are inverse pts. on a', b'. Hence P, Q are inverse pts. on a, b.

Ex. 2. Let the ⊙s be c_1, c_2, c_3, c_4. Let c_1, c_2 cut at O, A. Invert w. r. to O. Then c'_1, c'_2 are ⊥ |s, $A'X'$, $A'Y'$, say. Also c_3, c_4 invert into ⊥ ⊙s which are ⊥ to c'_1 and c'_2. Hence A' is the centre of c'_3 and of c'_4. Hence if r'_3, r'_4 are the radii of c'_3, c'_4, then $r'^2_3 + r'^2_4 = d^2 = A'A'^2 = 0$. Let P_1 be the inverse of P w. r. to c_1, P_2 of P_1 w. r. to c_2, P_3 of P_2 w. r. to c_3 and P_4 of P_3 w. r. to c_4. Then P'_1 is the reflexion of P' in c'_1, P'_2 is the reflexion P'_1 in c'_2, P'_3 is the inverse of P'_2 w. r. to c'_3 and P'_4 of P'_3 w. r. to c'_4. Now P', A', P'_2 are collinear; for $\angle P'_2A'P' = 2\,Y'A'P'_1 + 2\,P'_1A'X' = 2\,Y'A'X' = 180°$. Hence P', A', P'_2, P'_3, P'_4 lie on the same |. Again $A'P'_3 \,.\, A'P'_2 = r'^2_3 = -r'^2_4 = -A'P'_4 \,.\, A'P'_3$; $\therefore A'P'_4 = -A'P'_2 = A'P'$. Hence P'_4 coincides with P'. Hence P_4 coincides with P; and hence the figure f_4 generated by P_4 coincides with the figure f generated by P.

Page 120. § 15. Ex. 1. Let B bisect AC. Invert w. r. to O on AC. Now if I is the pt. at infinity on AC, (AC, BI) is hc, $\therefore (A'C', B'I')$ is hc. But I' is O, $\therefore (OB', A'C')$ is hc.

Ex. 2. Let the | l through A cut the ⊙s p, q, r, s through A at P, Q, R, S. Invert w. r. to A. Then p, q, r, s invert into |s p', q', r', s' through B'. Now p', q', r', s' meet at the same angles as p, q, r, s and hence form a hc pencil. l inverts into itself. Hence $PQRS$ inverts into the section $P'Q'R'S'$ of the hc pencil $(p'q'r's')$ by l. Hence $(P'Q'R'S')$ is hc, $\therefore (PQRS)$ is hc.

End of Chapter IX

Ex. 1. Taking O within ABC, we have

$$\angle BOC = OBA + OAB + OCA + OAC$$
$$= OAB + OAC + OA'B' + OA'C' = A + A'.$$

So for B and C. Similarly for other positions of O.

If the angles A, B, C, A', B', C' are given, we are given $\angle BOC$ and COA. Hence O is the other intn of the arcs on BC containing the given angle BOC and on CA containing the given angle COA.

Conversely with the pt. O so determined, invert ABC into $P'Q'R'$. Then by hyp. $\angle BOC = A + A'$, $\therefore$ $OBA + OAB + OCA + OAC = A + A'$, $\therefore$ $OBA + OCA = A'$. Also $OBA + OCA = OP'Q' + OP'R' = P'$, $\therefore$ $P' = A'$; so $Q' = B'$, $R' = C'$.

Ex. 2. Invert BCD w. r. to A into $B'C'D'$ and ACD w. r. to B into $A''C''D''$. Then

$$B'C' = k.BC/AB.AC,\ C'D' = k.CD/AC.AD,$$
$$\therefore\ B'C'/C'D' = BC.AD/CD.AB;\ \text{and so on.}$$
$$\therefore\ B'C' : C'D' : D'B' :: BC.AD : CD.AB : DB.AC.$$

So $$C''D'' : D''A'' : A''C'' :: CD.AB : DA.BC : AC.BD,$$
$$\therefore\ B'C' : C'D' : D'B' :: D''A'' : C''D'' : A''C''.$$

Hence the Δs $B'C'D'$ and $D''A''C''$ are similar; and so on.

Ex. 3. Let O be the pt. and $ABC\ldots$ the polygon. Invert w. r. to O. Then A', B', C',... are collinear. Hence (see p. 111, Ex. 3)

$$\frac{AB}{p} + \frac{BC}{q} + \ldots = \frac{A'B'}{p'} + \frac{B'C'}{p'} + \ldots = 0.$$

Ex. 4. Invert w. r. to A. Then the | $B'C' \perp D'E'$ and $B'D' \perp C'E'$. Hence B' is the orthocentre of Δ $C'D'E'$. Hence $B'E' \perp C'D'$. Hence in the given figure $ABE \perp ACD$.

Ex. 5. Invert w. r. to the common pt. and we get three |s. Four ⊙s can be drawn to touch these |s. Hence four ⊙s can be drawn to touch the given ⊙s.

Ex. 6. By p. 104, the two pts. are the intns X, Y of the three ⊙s of s. of the given ⊙s c_1, c_2, c_3. Let A, B be the

int^ns of c_1, c_2; and C, D of c_2, c_3. Then the ⊙ of s. (1, 2) of c_1, c_2 passes through A, B; so (2, 3) passes through C, D. Hence if AB, CD meet at R, then R is on the r. a. XY of (1, 2) and (2, 3) since $RA \cdot RB = RC \cdot RD$ from c_2. Also $RX \cdot RY = RA \cdot RB$ from (1, 2). But R is the centre and $RA \cdot RB$ the square of the radius of the radical ⊙. Hence X, Y are inverse w. r. to the radical ⊙.

Ex. 7. For if we invert w. r. to S, the ⊙ of s. inverts into a | which is still coaxal with the ⊙s.

Ex. 8. Let CD pass through S. Invert w. r. to S, taking C, D as inverse pts. Then the ⊙s invert into themselves, A, B inverting into A, B. Hence Δs SBC and SDB are similar. Hence $BC:BD::SB:SD$. So $AC:AD::SA:SD$, $\therefore BC:CA::BD:DA$, since $SB = SA$.

Ex. 9. Invert w. r. to a limiting pt. of the ⊙ and the |. Then the ⊙ and the | become concentric ⊙s. Hence the variable ⊙ will touch a fixed concentric ⊙. Hence in the given figure the variable ⊙ will touch a fixed coaxal.

Ex. 10. c_3 is the inverse of c_1 w. r. to c_2; hence c_1, c_2, c_3 are coaxal. Invert w. r. to a limiting pt. Then we get three concentric ⊙s with radii a, b, c, say. Since c'_3 is the inverse of c'_1 w. r. to c'_2, we have $b^2 = ac$ (for $OB^2 = OA \cdot OC$); so $c^2 = ab$, $\therefore b^2c^2 = a^2bc$, $\therefore a^2 = bc$, $\therefore c'_2$ is the inverse of c'_3 w. r. to c'_1. Hence c_2 is the inverse of c_3 w. r. to c_1.

Ex. 11. Invert w. r. to L. Then the ⊙s become concentric, and ⊙ LPQ becomes a | touching one ⊙ at P' and cutting the other at Q'. Hence $P'Q'$ is const. But $P'Q' = k \cdot PQ/LP \cdot LQ$, $\therefore LP \cdot LQ/PQ$ is const.

Ex. 12. By p. 119, Ex. 1, if the pts. are P, Q and P', Q', PQ', QP' pass through S and PP', QQ' through S'.

Ex. 13. Invert c_1 and c_2 w. r. to one of their int^ns. Then c'_1 and c'_2 are ⊥ |s. Hence P'_1 is the reflexion of P' in c'_1 and P'_2 is the reflexion P' in c'_2. Now obviously the reflexion of P'_1 in c'_2 is the same as the reflexion of P'_2 in c'_1. Hence in the given figure the inverse of f_1 w. r. to c_2 and of f_2 w. r. to c_1 coincide.

Ex. 14. Invert w. r. to O. Then the $\odot^s$ become equal $\odot^s$ through A', B'. Hence $A'B'$ bisects the angles between the $\odot^s$. Hence in the original figure the $\odot$ OAB bisects the angles between the $\odot^s$.

Ex. 15. Invert w. r. to O. Then we get the | $X'Y'Z'$ cutting the $|^s$ $B'C'$, $C'A'$, $A'B'$ at X', Y', Z'.

Hence $C'X' . B'Z' . A'Y' = -X'B' . Z'A' . Y'C'$.

But $C'X' = k . CX/OC . OX$, and so on.

Hence $CX . BZ . AY = -XB . ZA . YC$.

Ex. 16. Invert w. r. to O. Then O inverts into the pt. I' at infinity on $P'Q'$. Hence $(I'R', P'Q')$ is h^c, $\therefore$ R' bisects the common tangent $P'Q'$. Hence R' lies on the r. a. Hence the locus of R is the $\odot$ through O, coaxal with the given $\odot^s$.

Ex. 17. Invert w. r. to any pt. on the $\perp$ $\odot$, r. Then r becomes the common diameter of a', b', c'. Obviously the $\odot$ coaxal with a', b' through P' touches c'. Hence the $\odot$ coaxal with a, b through P touches c.

Ex. 18. Invert w. r. to A. Then the given $\odot^s$ b, c invert into the $\parallel$ $|^s$ b', c', outside which A lies; the variable $\odot$ x becomes x', a $\odot$ touching b', c'; the inverse of A w. r. to x becomes the centre of x'. Now the locus of the centre of x' is a | d' half-way between b' and c'. Hence the locus of the inverse of A is a $\odot$ d touching b, c at A. Let the $\perp$ from A to b', c', d' cut them at B', C', D'. Then $2/d = 1/b + 1/c$ if $2\,AD' = AB' + AC'$ (by p. 110); and this is true.

Ex. 19. Invert w. r. to an int^n of the given $\odot^s$ a, b. Then the variable $\odot^s$ x, y become $\odot^s$ x', y' touching the $|^s$ a', b' in the same angle. Hence the locus of the limiting pts. is a bisector of the $|^s$. Hence in the given figure the locus is the two $\odot^s$ coaxal with the given $\odot^s$ and bisecting the angles between them.

Ex. 20. Invert w. r. to A. Then c_1, c_2 becomes $|^s$, and c_3 becomes a $\odot$ with its centre at the int^n of the $|^s$. Hence the $|^s$ BC and $B'C'$ are $\parallel$. Hence in the given figure the $\odot^s$ ABC, $AB'C'$ touch at A.

CHAPTER X

Page 122. Ex. $OC \parallel BB'$ since $AC:CB'::AO:OB$, and $OC' \parallel AA'$ since $A'C':C'B::AO:OB$. But $AA' \parallel BB'$ since the $\perp^s$ from B and B' on AA' are equal. Hence OCC' is a $|$. Now O is a fixed pt., and C moves on a fixed $\odot$ through O since $CD = OD$. Hence C' will move on the inverse of the locus of C, i.e. on a $|$, if $OC.OC'$ is const. But $OC:BB'::AO:AB$ and $OC':AA'::BO:BA$. Hence $OC.OC'$ is const. if $AA'.BB'$ is const. Now since $\angle ABA' = AB'A'$, $ABB'A'$ is cyclic. Hence, by Ptolemy's theorem, $AB.B'A' + AA'.BB' = BA'.B'A$. Hence

$$AA'.BB' \text{ is const.}$$

Page 124. Ex. 1. Since the order is not $ACBD$,

$$AD.BC + AC.BD > AB.CD.$$

Ex. 2. $AB.PC + AC.PB = AP.BC$, $\therefore$ $PC + PB = PA$.

Ex. 3. Invert w. r. to A. Then $AP = k/AP'$; $BC = kB'C'/AB'.AC'$, and so on. Hence

$$(k^2/P'A^2).(k^2B'C'^2/B'A^2.C'A^2)$$
$$= (k^2P'B'^2/P'A^2.B'A^2).(k^2/C'A^2)$$
$$+ (k^2P'C'^2/P'A^2.C'A^2).(k^2/B'A^2),$$

$\therefore$ $B'C'^2 = P'B'^2 + P'C'^2$. Hence the locus of P' is the $\odot$ through B', $C' \perp B'C'$. Hence the locus of P is the $\odot$ through B, $C \perp$ the $\odot$ ABC.

Page 126. Ex. Invert w. r. to the polar $\odot$. Then the $\odot$ on AH as diameter becomes the $|$ through the inverse of A (i.e. through D) $\perp$ HA, i.e. becomes BC; so for BH, CH. Hence the four touching $\odot^s$ become the inscribed and escribed $\odot^s$ of ABC. But these touch the $\odot$ DEF. Hence in the given figure the four $\odot^s$ touch the $\odot$ ABC.

Page 127. § 5. Ex. 1. As in the text,

$$I_1H^2 = 2I_1N^2 + \tfrac{1}{2}OH^2 - OI_1^2$$
$$= 2(\tfrac{1}{2}R + r_1)^2 + \tfrac{1}{2}(R^2 + 2\rho^2) - R^2 - 2Rr_1$$
$$= \tfrac{1}{2}R^2 + 2Rr_1 + 2r_1^2 + \tfrac{1}{2}R^2 + \rho^2 - R^2 - 2Rr_1 = 2r_1^2 + \rho^2.$$

Ex. 2. Since $IN (= \tfrac{1}{2}R \backsim r)$ is known and I is fixed, the locus of N is a $\odot$.

Page 127. § 6. Ex. 1. To describe a ⊙ through A to touch the ⊙ b and to be ⊥ ⊙ c, invert w. r. to A. Then we want a | x' to touch b' and to be ⊥ c'; i.e. we have to draw a tangent to b' from the centre of c'. Inverting back, we get two solutions.

Ex. 2. To describe a ⊙ to touch the | l at A and to touch the ⊙ b, invert w. r. to A. Then we want to draw a | ‖ to l to touch the ⊙ b'. Hence, inverting back, we get two solutions.

Page 129. Ex. 1. To describe a ⊙ x to touch the ⊙ˢ a, b and to be ⊥ the ⊙ c, invert w. r. to c. Then x and c invert into themselves and a inverts into a'. Hence x touches a, b, a'. Also of the eight ⊙ˢ which touch a, a', b, we see by § 8 that half will be ⊥ to s (i. e. c). Hence there are four solutions.

Ex. 2. If the ⊙ x is isogonal to the ⊙ˢ a, b, it passes (p. 119) through a pair of inverse pts. on the ⊙ˢ, i.e. is ⊥ to a ⊙ of s. of a, b, say, s. So x is ⊥ to a ⊙ of s. of b, c, say, s'. Hence x is ⊥ s and s', and ∴ belongs to the coaxal system ⊥ s and s'.

Ex. 3. Suppose the ⊙ x, with centre X, cuts the ⊙ a, with centre A, at the end of the diameter MN of a. Then $AX \perp MN$, since $MA = AN$. Hence $XM^2 = XA^2 + AM^2$, $\therefore XA^2 = x^2 - a^2$; so $XB^2 = x^2 - b^2$, $\therefore XA^2 - XB^2 = b^2 - a^2$ and is known. Hence X lies on a certain | (see p. 34). So X lies on another |; and hence is known. Then $x^2 = XA^2 + a^2$ and is known.

Conversely, describe a ⊙ with the centre X, so determined and with radius $x = \sqrt{XA^2 + a^2}$; and let it cut a at M, N. Then $x^2 = XM^2 = XA^2 + a^2 = XA^2 + AM^2$; $\therefore \angle XAM = 90°$, $= XAN$ similarly. Hence MN passes through A. But, by the first |, $XB^2 = XA^2 + a^2 - b^2 = x^2 - b^2$, $\therefore x^2 = XB^2 + b^2$. Hence the same is true for the ⊙ b; and similarly for the ⊙ c.

Page 130. Ex. First consider three ⊙ˢ c_1, c_2, c_3; let t_1 be

a common tangent of c_1, c_3 and t_2 of c_2, c_3. Then $t_1{}^2/r_1r_3$ is unaltered by inversion; and so is $t_2{}^2/r_2r_3$. Hence $t_1{}^2r_2/t_2{}^2r_1$ is unaltered by inversion whatever value r_3 has. Now make $r_3 = 0$.

Page 131. Ex. We can take the pt. as a ⊙ 4 of zero radius. Then $12.34 \pm 14.23 \pm 13.24 = 0$ gives

$$12.t_3 \pm 23.t_1 \pm 13.t_2 = 0.$$

End of Chapter X

Ex. 1. Invert w. r. to A. Then $AB = k/AB'$ and $BD = k.B'D'/AB'.AD'$; and so on. Hence we have to prove that for the collinear pts. B', C', D',

$$(k.B'D'/AB'.AD').(k.C'D'/AC'.AD').(k.B'C'/A\,'.AC')$$
$$+(k/AD').(k.B'D'/AB'.AD').(k/AB')$$
$$=(k.B'C'/AB'.AC').(k/AC').(k/AB')$$
$$+(k\,C'D'/AC'.AD').(k/AD').(k/AC') = 0,$$

or dividing by k^3 and multiplying by

$$AB'^2.AC'^2.AD'^2,$$

that

$$B'D'.C'D'.B'C' + B'D'.AC'^2 = B'C'.AD'^2 + C'D'.AB'^2,$$

or

$$B'A^2.C'D' + C'A^2.D'B' + D'A^2.B'C' + C'D'.D'B'.B'C' = 0.$$

Now see p. 33.

Ex. 2. I is the orthocentre of $I_1I_2I_3$. Hence the ⊙ ABC is the N.P.C. of each of the Δ^s II_1I_2, II_2I_3, II_3I_1, $I_1I_2I_3$.

Ex. 3. We have seen (p. 127, Ex. 2) that the locus of N is a ⊙. Now O is fixed and $OH = 2.ON$. Hence the locus of H is a ⊙.

Ex. 4. A ⊙ touches i_1, i_2 at P, Q; hence P, Q are inverse pts. on the ⊙^{s} i_1, i_2. Hence PQ passes through the external c. of s. of i_1 and i_2 which is the int^n of I_1I_2 with the common tangent AB.

Ex. 5. By p. 130, $(12)^2 : (1'2')^2 :: r_1r_2 : r'_1r'_2 :: t_1t_2 : t'_1t'_2$ (by similar Δ^s) $:: k^2/t'_1t'_2 : t'_1t'_2 :: k^2 : t'_1{}^2t'_2{}^2$.

Ex. 6. To describe a ⊙ to pass through A and to touch the ⊙ b and to have its centre on the | l, take A' the reflexion of A in l and describe a ⊙ (p. 127) to pass through A, A' and to touch b.

CHAPTER XI

Page 135. Ex. 1. Let D bisect the supplementary arc AB. Then $AP \,.\, BD + BP \,.\, AD = AB \,.\, PD$. Hence $AP + BP = AB \,.\, PD \,/\, BD$ and is greatest when PD is greatest, i. e. when PD is CD, i. e. when P is at C.

Ex. 2. Let the polygon be $ABCD \ldots$. If $AB \neq BC$ take B' bisecting the arc ABC. Then $AB' + B'C > AB + BC$, $\therefore \; AB' + B'C + CD + \ldots > AB + BC + CD + \ldots$. In this way the polygon can be increased in perimeter by making two consecutive sides equal unless all the sides are equal. Hence the perimeter is greatest when all the sides are equal.

Page 136. Ex. 1. With the figure on p. 135, we want PP' of given length, $2\,l$, i. e. $O'N$ of length l. Hence, with O' as centre and l as radius, describe a ⊙ to cut again, at N, the ⊙ on OO' as diameter. Through A draw $PP' \parallel O'N$. Let ON (which is $\perp$ NO' and $\therefore$ $\perp$ PP') meet PP' at M. Draw $O'M' \perp PP'$. Then, as in the text, $PP' = 2 \,.\, O'N = 2\,l$.

Ex. 2. Draw (in the figure on p. 135) $BR \perp PP'$. Then $\Delta BPP' = \frac{1}{2} BR \,.\, PP'$. Now PP' is greatest when $PP' \parallel OO'$, i. e. $\perp AB$. Then BR coincides with BA; and this is the greatest value of BR, for $BR < AB$ unless R coincides with A.

Ex. 3. Suppose that PQ, QR, RP have to pass through the fixed pts. C, A, B. Then, since the form of PQR is given, the angles P and Q are known; hence P and Q lie on known ⊙s passing through C. Hence PQ is greatest when PQ is $\parallel$ the | of centres O, O' of these arcs. Hence (since the triangles such as PQR are all similar) the area of $\Delta\, PQR$ is greatest when PQ is $\parallel$ OO'.

Ex. 4. By Ex. 1 we construct PQ of given size.

Page 137. Ex. 1. Take the $\Delta\, P'Q'R'$ which is of right shape and of right area. Obtain the pt. O' as the other int^n of arcs on $P'Q'$ and $Q'R'$ containing angles $180° - C$ and $180° - A$. Then, in the figure on p. 136, we want $OP = O'P'$; hence with O as centre we describe a circle with $O'P'$ as radius. This gives two possible positions of P. Taking P at either of these pts., we can construct Q and R, since the angles OPQ and OPR are known.

Ex. 2. If $P = A$, $Q = B$, $R = C$, then $\angle AOB = C + R = 2C$; and so on. Hence O is the circumcentre; for the circumcentre is also the other int^n of arcs described on AB, BC inwards containing angles $2C$ and $2A$.

Page 138. Ex. 1. Take any pt. P on l. Then

$$BP - AP = BP - A'P < A'B < QB - QA' < QB - QA.$$

Ex. 2. Let PQR be inscribed in ABC, so that P is on BC, Q on CA, R on AB. If PQ, RQ are not equally inclined to CA, take Q' on CA such that PQ', RQ' are equally inclined. Then $PQ' + Q'R < PQ + QR$, $\therefore\ PQ' + Q'R + RP < PQ + QR + RP$. Hence, unless BC, CA, AB bisect externally the angles P, Q, R, we can decrease the perimeter of PQR. Hence the $\Delta\, PQR$ of least perimeter has its angles so bisected. Now see p. 22, Ex. 1.

Ex. 3. For, if not, we can, as in Ex. 2, decrease the perimeter by making two of the unequal angles equal.

Ex. 4. Let P, Q, R be three consecutive vertices; and let Q move, all the other vertices remaining fixed. Then as Q moves, the only part of the area which alters is PRQ. Hence PRQ is const. Hence the locus of Q is a | || PR. Hence $PQ + QR$ is least when PQ, RQ make equal angles with the locus, i. e. with PR; i. e. when $PQ = QR$. Hence, unless the polygon is equilateral, we can decrease the perimeter by making two consecutive unequal sides equal. Hence the perimeter is least when the polygon is equilateral.

Page 139. Ex. 1. Let PQ, QR be consecutive sides which are not equal. Take Q' such that

$$PQ + QR = PQ' + Q'R \text{ where } PQ' = Q'R.$$

Then area $PQ'R >$ area PQR. Hence the area of the polygon ... $PQ'R$ ... $>$ ... PQR .... Hence if two consecutive sides are not equal we can increase the area by making these equal. Hence the equilateral polygon has the greatest area.

Ex. 2. For clearness consider a square $ABCD$ and a regular pentagon $A'B'C'D'E'$ of the same perimeter. Then we may consider the square to be an irregular pentagon with zero side DE. Hence area $A'B'C'D'E' >$ area $ABCDE$ $>$ area $ABCD$.

So other cases may be dealt with.

Ex. 3. Given the base BC and the angle A, the locus of A is an arc of a ⊙. From A draw $AN \perp BC$. Then the area is greatest when AN is greatest, i.e. when A bisects the arc, i.e. when $AB = AC$.

Ex. 4. If two consecutive sides PQ, QR are unequal, we can (as in Ex. 3) increase the area PQR and ∴ the area of the polygon by taking Q' for Q if Q' bisects the arc PQR.

Page 141. Ex. For brevity take a pentagon $ABCDE$. Let $A'B'C'D'E'$ be the regular cyclic pentagon with the same area. We have to show that $AB + BC + \dots + EA > A'B' + B'C' + \dots + E'A'$. Take the regular pentagon $A''B''C''D''E''$ with the same perimeter as $ABCDE$; so that $AB + \dots = A''B'' + \dots$. Then by § 8, end, area $A''B''C'' \dots >$ area $ABC \dots$. But area $ABC \dots =$ area $A'B'C' \dots$, ∴ area $A''B''C'' \dots >$ area $A'B'C' \dots$. But $A''B''C'' \dots$ and $A'B'C' \dots$ are similar figures, ∴ $A''B'' + \dots > A'B' + \dots$. But $A''B'' + \dots = AB + \dots$, ∴ $AB + BC + \dots + EA > A'B' + B'C' + \dots + E'A'$.

Page 142. Ex. 1. Since $PL + PN$ is const., $PL^2 + PN^2$ is least when $PL = PN$. So $PM^2 + PR^2$ is least when $PM = PR$. Hence $PL^2 + PM^2 + PN^2 + PR^2$ is least when $PL = PM = PN = PR$, i.e. when P is at the centre of the square.

Ex. 2. Let PL, PM, PN be the ⊥s from P on the sides BC, CA, AB of a Δ. First take PL const. Then P moves

on a $\parallel$, XY, to BC. Now $PN = XP \sin X \propto XP$; so $PM \propto PY$. Hence $PM \cdot PN \propto PX \cdot PY$; and this is greatest when $PX = PY$ since $PX + PY$ is given, i.e. when P lies on the median AA'. Hence unless P lies on AA', we can increase $PL \cdot PM \cdot PN$ by taking P' on AA', keeping PL const. Hence when $PL \cdot PM \cdot PN$ is greatest, P must be on each median, i.e. at the centroid.

Ex. 3. Let the $\parallel^m$ be $OPRQ$. Then the area, viz. $OP \cdot OQ \sin O$, is greatest when $OP \cdot OQ$ is greatest. But the perimeter is given, i.e. $OP + OQ$ is given. Hence the area is greatest when $OP = OQ$, i.e. when the $\parallel^m$ is equilateral.

Ex. 4. We have seen in Ex. 3 that if the $\parallel^m$ is not equilateral we can increase the area by making it so. Now draw $QX \perp OP$. Then area $= QX \cdot OP < OQ \cdot OP < OP^2$ (since $OP = OQ$); i.e. less than the area of the square on OP; i.e. less than the area of the square with the same perimeter.

Ex. 5. Area $PQBR = PR \cdot PQ \sin B \propto PR \cdot PQ$

$$\propto (PA \sin A / \sin B) \cdot (PC \sin C / \sin B) \propto PA \cdot PC$$

which is greatest when $PA = PC$ since $PA + PC$ is const.

Ex. 6. Let P' be the position of P when $SP = PT$. Let QP cut $S'T'$ at P''; draw $P''R'' \parallel OA$. Then by Ex. 5, area $OQ'P'R' > OQP''R'' > OQPR$.

Ex. 7. Let the sides of the rectangle be x, y. Then we are given xy and we want $x^2 + y^2$ least. But $x^2 + y^2 = 2xy + (x-y)^2$ and is least when $x = y$; i.e. in the case of a square.

End of Chapter XI

Ex. 1. With the figure of p. 135, we have $MA : AM'$ given. In OO' take C so that $OC : CO' :: MA : AM'$; then $CA \parallel OM$ and $\therefore \perp PP'$. Hence PP' must be drawn $\perp CA$.

Ex. 2. With the figure of p. 136, since the Δ^s OQ_1R_1 and OQ_2R_2 are similar, $\therefore OQ_1 : OR_1 :: OQ_2 : OR_2$ and $\angle Q_1OR_1 = Q_2OR_2$. Hence $OQ_1 : OQ_2 :: OR_1 : OR_2$ and $\angle Q_1OQ_2 = R_1OR_2$. Hence the Δ^s Q_1OQ_2 and R_1OR_2 are similar,

$\therefore\ Q_1Q_2 : R_1R_2 :: OQ_2 : OR_2 :: Q_2Q_3 : R_2R_3$, similarly. Hence $Q_1Q_2 : Q_2Q_3 :: R_1R_2 : R_2R_3$; and so on.

Ex. 3. The meaning is that BC bisects the angle MLR externally, and so on; i.e. that EL bisects it internally. Now $\angle MLE = MCE$ since $MCLE$ is cyclic

$= RBE = RLE$ since $RBLE$ is cyclic.

Ex. 4. The direction of PQ being given, its length is given. Hence $AP+QB$ has to be least. Complete the $\|^{\text{m}}$ $BQPC$. Then C is a given pt., since BC is given in magnitude and direction. Also $QB = PC$; hence $AP+PC$ must be least. Hence A, P, C must be collinear. Hence AC cuts l in the required pt. P; and now Q is known.

Ex. 5. Let the diagonals AC, BD cut at E. Let $AE=x$, $BE = y$, $CE = z$, $DE = u$. Then

$$\begin{aligned}\text{area} &= \tfrac{1}{2}xy\sin E+\tfrac{1}{2}yz\sin E+\tfrac{1}{2}zu\sin E+\tfrac{1}{2}ux\sin E\\ &= \tfrac{1}{2}(x+z)(y+u)\sin E = \tfrac{1}{2}AC\,.\,BD\sin E,\end{aligned}$$

which is greatest when $\sin E$ is greatest, i.e. when $E = 90°$.

Ex. 6. Let $OPQR$ be the $\|^{\text{m}}$. First, keeping the angle O unchanged, take $OP' = OR'$ so that $OP'^2 = OP\,.\,OR$. Then the area is unchanged; but $OP+OR$ is decreased, for $(OP'+OR')^2 = 4\,OP'^2 = 4\,OP\,.\,OR = (OP+OR)^2-(OP-OR)^2 < (OP+OR)^2$.

Next take a square $OP''Q''R''$ equal in area to $OP'Q'R'$, so that $OP''^2 = OP'^2\sin P'OR'$; hence $OP'' < OP'$. Hence $OP''+OR'' < OP'+OR' < OP+OQ$. Hence the square has the least perimeter.

Ex. 7. Let the Δ^{s} be PA_1A_2, PB_1B_2, PC_1C_2. Since these triangles are similar, their areas are as the squares of corresponding sides. Hence $PA_1A_2 : PB_1B_2 : PC_1C_2 :: A_1A_2{}^2 : A_2C^2 : BA_1{}^2$. Hence we want $BA_1{}^2+A_1A_2{}^2+A_2C^2$ least, given $BA_1+A_1A_2+A_2C$. Hence $BA_1 = A_1A_2 = A_2C$. Hence $C_2P = PB_1$; i.e. P is on AA'. So P is on BB' and $\therefore$ at G.

Ex. 8. Let $PQRS$ be the rectangle. Then by symmetry the centre O of the $\odot$ bisects PQ. Hence the area $= PQ\,.\,QR = 2\,OQ\,.\,QR$. Hence we want $OQ\,.\,QR$ greatest given

$OQ^2+QR^2=a^2$. But $2OQ.QR=OQ^2+QR^2-(OQ-QR)^2$ which is greatest when $OQ=QR$, $\therefore 2OQ^2=a^2$. Hence $PQ=2OQ$ is the diagonal of a square of side a.

Ex. 9. Since P moves on the arc, the angle P is const. Hence $AP.PB \propto AP.PB\sin P \propto$ area APB which is greatest when P bisects the arc (by p. 139, Ex. 3).

Ex. 10. If θ is the angle between the half-diagonals a, b, we want $2ab\sin\theta$ greatest. Hence $\theta=90°$, i.e. the diagonals are $\perp$.

Ex. 11. Area $=PQ.PR=(PA\sin A/\sin C).(BP\sin B)$ $\propto PA.PB$ which is greatest when $PA=PB$, since $PA+PB$ is const.

CHAPTER XII

Page 148. Ex. 1. Let the sides QR, RP, PQ of the required Δ pass through A, B, C. Let QR vary while the $|^s$ PQ and PR remain fixed. Then, unless $QA=AR$, we can decrease area PQR by making $QA=AR$. Hence the triangle of least area must have its sides bisected at A, and so at B, C. Also since $RA:AQ::RB:BP$, $PQ \parallel AB$; so for QR, RP.

Ex. 2. Take $P'Q'$ through A consecutive to PQ. Then area $AP+AQ$ has a critical value when area $AP+AQ$ $=AP'+AQ'$; i.e. when area $APP'=AQQ'$. With A as centre describe $\odot^s$ with radii AP and AP' to cut AP' at R and AP at R'. Then area APP' lies between area APR and $AP'R'$, i.e. between $\frac{1}{2}AP^2\sin A$ and $\frac{1}{2}AP'^2\sin A$ and hence is ult^{ly} equal to $\frac{1}{2}AP^2\sin A$; so for area AQQ'. Hence we have $\frac{1}{2}AP^2\sin A=\frac{1}{2}AQ^2\sin A$, $\therefore AP=AQ$. Now see p. 143, end Ex. 1.

Ex. 3. We must have $PA.AQ=P'A.AQ'$. Hence P, Q, P', Q' are concyclic. Hence ult^{ly} when P and P' coincide, the tangents to the given $\odot^s$ at P and Q touch a $\odot$ and are $\therefore$ equally inclined to PQ. Hence the tangents at A are equally inclined to PQ; i.e. PQ bisects the angle between the tangents at A. Also this critical value is the greatest

value. For when PQ coincides with either tangent at A, $PA \,.\, AQ$ is zero; hence the above solution gives a unique critical value lying between two absolute minima.

Page 149. Ex. 1. Let PQ touch at R. Then area CPQ $[= \frac{1}{2} CR \,.\, PQ]$ is least when PQ is least. Now, as in the text, area CPQ is critical when $PR = RQ$. Also this critical area CP_0Q_0 is the least. For let $P'Q'$ be a tangent near to P_0Q_0. Through R_0 draw $P''Q'' \parallel P'Q'$. Then area $CP'Q' > CP''Q'' > CP_0Q_0$ (by § 2). Hence we have a unique critical value lying between two greater values.

Ex. 2. With the figure of the text, let p and p' be the $\perp^s$ from C on PQ and $P'Q'$. Then since the area is const., $p \,.\, PQ = p' \,.\, P'Q'$; and since PQ has a critical value, $PQ = P'Q'$, $\therefore p = p'$. Hence PQ and $P'Q'$ ult^{ly} touch a $\odot$ with C as centre and p as radius. Hence, as in the text, PQ touches this $\odot$ at the centre of PQ, and hence is equally inclined to CA and CB. Again, since $CP \,.\, CQ$ is const., if we take P at C, so that $CP = 0$, we get $CQ = \infty$ and hence $PQ = \infty$; so if Q is at C. Hence the critical value is unique and lies between two greater values and is $\therefore$ the least value.

Page 150. Ex. 1. Let $P'Q'$ be a consecutive position of PQ, then $P'Q' = PQ$. Also $P'Q' \parallel PQ$, $\therefore PP' \parallel QQ'$. Hence ultimately the tangents at P and Q are $\parallel$.

Ex. 2. Let $AN = x$, $PN = y$, $AN' = x+p$, $P'N' = y-q$. Then $AN \,.\, PN = AN' \,.\, P'N'$ gives $xy = (x+p)(y-q)$ or $xy = xy - xq + py - pq$ or $xq = py - pq$ or $x/y = p/q - p/y = p/q$ ultimately when $p = 0$. Draw $P'M \perp PN$. Then $p = P'M$, $q = PM$. Hence $AN/PN = P'M/PM$, $\therefore \angle PAN = PP'M$. Hence ult^{ly} PA and the tangent at P are equally inclined to BC.

Ex. 3. Take a consecutive position $P'Q'$ of PQ. Then $AQ'^2 - Q'P'^2 = AQ^2 - QP^2$, $\therefore AQ'^2 - AQ^2 = Q'P'^2 - QP^2$. Now $AQ'^2 - AQ^2 = (AQ' + AQ)(AQ' - AQ) = 2AQ \,.\, QQ'$ ult^{ly} since ult^{ly} $AQ' = AQ$. Also $Q'P'^2 - QP^2 = (Q'P' - QP)$. $(Q'P' + QP) = P'M \,.\, 2QP$, ult^{ly} if $PM \perp P'Q'$. Hence

$AQ \,.\, QQ' = P'M \,.\, QP$ or $AQ/QP = P'M/QQ' = P'M/PM$. Hence $\angle BAP = PP'M = B - 90°$.

Page 151. § **5. Ex. 1.** Take a consecutive position, P', of P. Then $AP - PB = AP' - P'B$. Hence we can describe a hyperbola with foci A, B to pass through P and P'. Hence the tangent of the curve at P (which is the limit of PP') is also the tangent of the hyperbola at P and $\therefore$ bisects the angle APB.

Ex. 2. Here $\angle AP'B = APB$. Hence a ⊙ can be drawn through A, B, P, P'. Hence ult^{ly} a ⊙ can be drawn through A, B to touch the curve at P. To prove that APB is greatest in this position, take any other pt. P' on the curve and let AP' cut the ⊙ at Q'. Then $\angle APB = AQ'B > AP'B$.

Ex. 3. Area $AP'B = APB \;\therefore\; PP'$ is $\parallel AB$; and hence the tangent at P is $\parallel AB$. Hence there are two positions of P, viz. the ends of the diameter $\perp AB$. Also each of these makes area APB a maximum. For taking P' on the same side of AB as P, we see that the altitude of $\Delta AP'B$ is less than that of APB. Of course the area is a minimum (viz. zero) at the pts. in which AB cuts the ⊙.

Page 151. § **6. Ex.** Let C bisect AB. Then $AP^2 + BP^2 = 2(PC^2 + AC^2)$. Hence $AP^2 + BP^2$ is least when CP is least. Now use the text.

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Ex. 1. Since AD is given, BC touches a ⊙ with centre A and radius AD. Also $AD \,.\, BC$ is to be least and AD is given; hence BC is to be least. Now see p. 149, Ex. 1.

Ex. 2. We may consider (as in p. 139, Ex. 2) a regular polygon of $n-1$ sides to be an irregular polygon of n sides, one side being zero. Now use p. 148, § 3.

Ex. 3. $QR = 2R \sin QPR$. Hence QR is greatest when the angle QPR is nearest $90°$. First suppose that the ⊙, b, on AB as diameter does not cut the given ⊙, a. Let AP (or BP) cut b at M; then $\angle APB < AMB < 90°$. Hence

QR is greatest when the angle APB is greatest. Hence by p. 151, § 5, Ex. 2, we have to find the pts. P_1, P_2 at which the ⊙s c_1, c_2 through A, B touch a. These are on opposite sides of AB; for, as the arcs on opposite sides of AB expand continuously, at one stage each passes from within a to without a, and at this pt. it touches a. Also at each of these pts. the angle APB is greatest. For if P is any pt. on a on the same side of AB as P_1, then AP (or BP) cuts c_1, at N, say. Then $\angle APB < ANB < AP_1B$; so for P_2. Also if AB cuts a at C, D, QR is zero when P is at C or D. Hence QR is a min. at C, max. at P_1, min. at D, and max. at P_2. Next suppose b cuts a at P_3, P_4. Then as the arc on AB expands, it becomes c_1 and then b; hence $\angle AP_1B > 90°$. Hence $Q_3R_3 = Q_4R_4 > Q_1R_1$. Also C, D, P_2 can be dealt with as before. Hence, in this case, QR is a min. at C, max. at P_3, min. at P_1, max. at P_4, min. at D, and max. at P_2.

Ex. 4. Let P' be a consecutive pt. to P. Then $AP : PB :: AP' : P'B$. Hence if we divide AB at Q and R in the ratio $AP : PB$, then P, P' lie on the ⊙, x, on QR as diameter (by p. 66). Hence ult^{ly} x touches l at P. Now A, B are inverse pts. w. r. to each of the ⊙s x, since (AB, QR) is h^c. Hence the ⊙s x form a coaxal system of which A, B are the limiting pts. Hence we have to draw a ⊙ of this system to touch l.

Ex. 5. When the ⊙ is divided most unequally by PQ, we have (taking the consecutive position $P'Q'$) area $APP' = AQQ'$. Hence as in p. 148, Ex. 2, $PA = AQ$.

Ex. 6. Take a consecutive position $P'Q'$ of PQ. Then $PO \cdot OQ = P'O \cdot OQ'$. Hence $PP'QQ'$ is cyclic. Hence if the given |s meet at C, then $CP \cdot CP' = CQ \cdot CQ'$ or ult^{ly} $CP^2 = CQ^2$, $\therefore$ $CP = CQ$. Hence PQ is $\perp$ the internal bisector of the angle PCQ. Least because infinite when $PQ \parallel$ to either |.

Ex. 7. Take a consecutive position P' of P. Then $AP^2 - PB^2 = AP'^2 - P'B^2$, $\therefore$ $PP' \perp AB$, i. e. ult^{ly} the tangent at P is $\perp AB$. Hence there are two solutions.

CHAPTER XIII

Page 154. Ex. 1. Let $AB = a$ and $CD = b$. Call the $\perp^s$ from P on AB and CD, x and y. Then if area $APB + CPD$ is given, we are given $ax + by$; and also $x + y$ if P is between the $|^s$. Hence x can be found. Hence the locus of P is a $\|$ to the given $|^s$. So other cases can be discussed.

If $a = b$ and P is between the $|^s$, we are given $ax + ay$ and $x + y$, i.e. $x + y$ only; hence P may be anywhere between the $|^s$. If $a = b$ and P is outside the $|^s$, we are given $ax - ay$ and $x - y$; hence as before.

Ex. 2. For clearness take the particular case when $ax + by - cz = k$. In the text, let $AB = a$, $CD = b$; then $ax + by = 2(OML) + 2(PML) = m + a'x'$ where m is a const. and $LM = a'$. Hence $ax + by - cz = k$ gives $m + a'x' - cz = k$ or $a'x' - cz = k'$. This can now be dealt with by the second part of the text. So other cases can be discussed.

Ex. 3. Let the $\perp^s$ be PL, PM on OA, OB. Produce LP to M', making $PM' = PM$. Then $M'L = M'P + PL = MP + PL$ which is given. Hence the locus of M' is a $\|$ to OA. Let this locus meet OB at C. Then $PM = PM'$; hence the locus of P is the bisector of the angle MCM'.

Ex. 4. Proceed as in Ex. 3 but draw PM' in the opposite direction. The solution is then the same.

Ex. 5. With the figure of p. 60, take the area (LAB) to be $+$, then (MAB) is $+$ and (NAB) is $-$; so if $(LA'B')$ is $+$, $(MA'B')$ is $+$ and $(NA'B')$ is $+$. Hence $(NAB) + (NA'B')$ algebraically $= (NA'B') - (NAB)$ arithmetically

$$= \tfrac{1}{2}(CA'B') - \tfrac{1}{2}(CAB) = \tfrac{1}{2}(ABA'B');$$

for $\quad CC' = 2 . NC' \;\therefore\; \Delta\, CA'B' = 2\,\Delta\, NA'B'$

and $\quad \Delta\, CAB = 2NAB.$

So $\quad (LA'B') + (LAB) = \tfrac{1}{2}(AA'B') + \tfrac{1}{2}(A'AB) = \tfrac{1}{2}(ABA'B')$
$\qquad = (MA'B' + MAB)$ similarly.

Hence $\quad (LAB) + (LA'B') = (MAB) + MA'B')$
$\qquad = (NAB) + (NA'B').$

Hence by the text L, M, N are collinear.

Ex. 6. In order to show that the centre O of the $\odot$ lies on LMN, we must show that $(OAB)+(OA'B')=\frac{1}{2}(ABA'B')$. Now if AB, BA', $A'B'$, $B'A$ touch at X, Y, Z, U,

$$\begin{aligned}&(OAB)+(OA'B')\\&=(OXA)+(OXB)+(OZA')+(OZB')\\&=\tfrac{1}{2}(OXA+OUA+OXB+OYB+OYA'+OZA'\\&\qquad+OZB'+OUB')\\&=\tfrac{1}{2}(ABA'B').\end{aligned}$$

Ex. 7. Let PL, PM be the $\perp^s$ on the $|^s$ OA, OB. Take a particular position P' of P. Then P lies on OP'. For if not let a $\parallel$ to OA through P cut OP' at P''. Draw the $\perp^s$ $P''L''$, $P''M''$. Then $P''L'':P''M''::P''L''/OP'':P''M''/OP''$ $::P'L'/OP':P'M'/OP'$ (by similar Δ^s)$::P'L':P'M'::PL$ $:PM$ (by hyp.)$::P''L'':PM$. Hence $P''M''=PM$. Hence PP'' is also $\parallel$ OB; i.e. P and P'' coincide. Hence P is on OP', which is $\therefore$ the locus.

Page 155. § 2. Ex. 1. It is sufficient to prove that $\angle SQ'Q=90°$. Now, since Δ^s SPQ, $S'P'Q'$ are similar, $SP:SQ::SP':SQ'$ and $\angle PSQ=P'SQ'$; hence $SP:SP'$ $::SQ:SQ'$ and $\angle PSP'=QSQ'$. Hence Δ^s SPP', SQQ' are similar. Hence $\angle SQ'Q=SP'P=90°$.

Ex. 2. If the base BC is given and the area ABC, the locus of A is a $|$ $\parallel$ BC. Also $A'G=\frac{1}{3}A'A$. Hence the locus of G is another $|$ $\parallel$ BC.

Page 155. § 3. Ex. 1. Draw OL, $OM \perp l$, m; then L, O, M are collinear and $OL=OM$. With O as centre and OL as radius describe a $\odot$ and let the other tangent from X touch at Z and cut m at Y'. Then $\angle XOY'=XOZ$ $+ZOY'=\frac{1}{2}(LOZ+ZOM)=90°=XOY$. Hence Y and Y' coincide; i.e. the envelope of XY is the $\odot$.

Ex. 2. Describe the $\odot$ escribed to XY and let it touch the $|^s$ OX, OY at A, B. Then $OA=OB=s=\frac{1}{2}(OX+OY$ $+XY)$ is given. Hence A and B are known. Hence the envelope of XY is this $\odot$.

Page 158. Ex. 1. I is the int^n of the $\perp^s$ to OX at A and OY at B. Also P is the proj^n of I on AB and IP is the

normal at P to the envelope of AB. Hence we have to prove that $IP \parallel OQ$. Now the Δ^s BOQ, AIP are congruent since $BO = AI$, $BQ = AP$, and $\angle OBQ = IAP$ (by $\parallel^s$). Hence $\angle OQB = IPA$; $\therefore$ $OQ \parallel IP$.

Ex. 2. Let the $\perp^s$ to the sides AB, AC at the fixed pts. L, M meet at I. Then the proj^n P of I on BC is the pt. of contact of BC with its envelope. Now the $\odot$ $ILAM$ is known since L, M are known pts. and A is a known angle. Let PI cut this $\odot$ again at O. Then O is a fixed pt. For $\angle LIO = B$ from the $\odot$ $LIPB$; hence the arc LO is known. Again $\angle OAB = OIL = B$; hence $OA \parallel BC$. Hence OP is known, being equal to the altitude AD. Hence the envelope of BC is the $\odot$ with centre O and radius OP.

Ex. 3. Let the centres of the $\odot^s$ l, m be L, M. Draw $A'B' \parallel AB$ through L and $A'C' \parallel$ to AC through M, B' and C' being on BC. Then $A'B'$, being $\parallel AB$ and at a distance l from it, is fixed to the Δ ABC; so $A'C'$. Hence the Δ ABC moves with the triangle $A'B'C'$ of which two sides now pass through the fixed pts. L, M. Hence by Ex. 2 the envelope of BC is a $\odot$.

Page 160. Ex. 1. Let $AB = 2a$ and $BX = x$. Then $AX.BX = c^2$ gives $(2a+x)x = c^2$, $\therefore$ $x+a = \sqrt{a^2+c^2}$ since $x+a$ is $+$. Hence $BX = \sqrt{a^2+c^2}-a$. Hence the construction—Bisect AB at C, draw $CD = c \perp AB$; then a $\odot$ with centre C and radius BD will cut AB (towards B) at X. For $CX = BD = \sqrt{a^2+c^2}$ and $BX = CX - a$.

Ex. 2. Let $AB = 2a$ and $XA = x$. Then $AB.XB = XA^2$ gives $2a(x+2a) = x^2$, $\therefore$ $XA = a + a\sqrt{5}$ (for $a - a\sqrt{5}$ is $-$). Hence the construction—At B draw $BD = a \perp AB$; then a $\odot$, with centre, C, on BA at a distance a beyond A and radius AD, will cut BA (beyond C) at the required pt. X. For $XA = XC + CA = AD + a = a\sqrt{5} + a$.

Ex. 3. Let the $|^s$ be x, y. Then we are given that $x^2 - y^2 = c^2$, $xy = m^2$, $\therefore$ $x^2y^2 = m^4$. Hence x^2 and $-y^2$ are the roots of the quadratic $z^2 - c^2z - m^4 = 0$ and can be

constructed by § 5. Suppose $x^2 = a^2$ and $-y^2 = -b^2$, $\therefore x = a, y = b$.

Page 161. Ex. 1. In the ΔABC, we are given $BC, B-C$, and $BA+AC$. Produce BA to D until $BD = BA+AC$, so that $AD = AC$. Then BD is known; hence the locus D is a $\odot$. Also $\angle BCD = C+ACD = C+CDA$ (since $AD = AC$) $=$ (also) $180° - B - CDA = \frac{1}{2}(C+CDA+180° - B - CDA) = 90° + \frac{1}{2}(C-B)$ which is known. Hence D is one of the int^ns^ of the above $\odot$ with the | CD drawn at the angle $90° + \frac{1}{2}(C-B)$ with CB. Then to get A, make $\angle DCA = BDC$.

Ex. 2. We are given BC, A, and $BA+AC$. Produce BA to D until $BD = BA+AC$. Then $A = ADC + ACD = 2ADC$. Hence $\angle ADC = \frac{1}{2}A$ and is known. Hence one locus of D is a known $\odot$. Also BD is known. Hence another locus of D is a $\odot$. D may be either int^n^ of these $\odot$s. Then to get A, make $\angle DCA = BDC$.

Ex. 3. Let ABC be the Δ. Then O is known. Also $OA' = \frac{1}{2}AH$ and is known. Hence, taking OA' in the same direction as AH, the line of B, C (viz. a | through A' $\perp$ OA') is known. Then a $\odot$ with centre O and radius R will cut this line at B, C.

Ex. 4. With the figure of p. 24, $XA' \parallel OA$, $\therefore \angle NXH = OAH = BAH - BAO = (90° - B) - (90° - C) = C - B$ and is known. Also N and H are known; hence X lies on a certain $\odot$. Also X lies on the given N.P.C. Hence X is either of the int^ns^ of these $\odot$s. Then $HA = 2 . HX$ gives A; and $HO = 2 . HN$ gives O. Also $R = 2n$. Hence A, H and the circum$\odot$ are known. Now see Ex. 3.

Page 163. Ex. 1. With the figure of p. 161, if $x : y :: bc + ad : ab + cd$ and yet $ABCD$ is not cyclic, let $A'B'C'D'$, with the same sides a, b, c, d, be cyclic, $\therefore x' : y' :: bc + ad : ab + cd$, $\therefore x' : x :: y' : y$. First suppose $x' > x$, $\therefore y' > y$. Then in the Δs BAD, $B'A'D'$ we have $BA, AD = B'A', A'D'$ and $B'D' > BD$, $\therefore A' > A$; so $B' > B$, $C' > C$, $D' > D$. Hence $A' + B' + C' + D' > A + B + C + D$; which is impossible for

each sum is 360°. Hence $x' \not> x$; so $x' \not< x$. Hence $x' = x$ and $\therefore$ $y' = y$. Hence $A'B'C'D'$ is congruent to $ABCD$ and $\therefore$ cyclic.

Ex. 2. Let the given angle be F. Describe the $\Delta\, ABE$ with $AB = a$, $BE = c$, and $ABE = F$. With A, E as centres and d, b as radii, describe $\odot^s$ cutting at D. Complete the $\|^m$ $DCBE$. Then $ABCD$ is the required quad. For $AB = a$, $BC = DE = b$, $CD = BE = c$, $DA = d$; and the angle between AB and $CD = ABE = F$.

Ex. 3. Let l, m, n, r bisect AB, BC, CD, DA $\perp^{ly}$. Take A_1, A_2 at random; and let the successive reflexions of A_1, A_2 in l and m and n and r be B_1, B_2 and C_1, C_2 and D_1, D_2 and A'_1, A'_2. Now B, C, D, A are the successive reflexions of A. Hence $AA_1 = BB_1 = CC_1 = DD_1 = AA'_1$. So $AA_2 = AA'_2$. Hence A is the int^n of the $\perp$ bisectors of $A_1 A'_1$ and $A_2 A'_2$.

Page 164. Ex. 1. Take $AX \perp$ and $= BD$. Join CX cutting $\perp^s$ to CX through B and D at M and N. Let these $\perp^s$ cut a $\|$ through A to CX at L and R. Then $LMNR$ is a square circumscribed to $ABCD$. By construction it is a rectangle; we must further prove that its sides are equal. Draw AY, $BZ \perp MN$, NR. Then in the Δ^s AXY, BDZ, we have $AX = BD$, $Y = Z = 90°$, and $\angle AXY = BDZ$ (since $AX \perp BD$ and $XY \perp DZ$). Hence $AY = BZ$, i.e. $LM = MN$.

Ex. 2. This is the same problem as in Ex. 1, taking D to coincide with A. Hence draw $AP =$ and $\perp AB$. Draw $\perp^s$ AZ, BY to CP and $AX \perp BY$. Then $AB \perp AP$ and $AX \perp AZ$, $\therefore \angle BAX = PAZ$. Hence $AB = AP$, $X = Z = 90°$ and $\angle BAX = PAZ$. Hence $AX = AZ$. Hence $AZYX$ is a square.

Page 165. Ex. 1. Produce $D'G'$ to H' and make the angle $G'F'B' = 180° - F'G'H' - B$, B' being on $G'H'$. Similarly construct $A'E'$; and let $A'E'$, $B'F'$ meet at C'. Then $\Delta\, A'B'C'$ is similar to ABC; for $B' = 180° - F'G'H'$

$-G'F'B' = B$; so $A' = A$; $\therefore$ $C' = C$. Now describe a figure $ABCDEFG$ similar to $A'B'C'D'E'F'G'$. Then $DEFG$ is the required figure. To do this, take F such that $BF : FC :: B'F' : F'C'$ and so on.

Ex. 2. $AD \,.\, BC = BE \,.\, CA = CF \,.\, AB$ $\therefore$ $BC \propto 1/AD$ and so on. To construct $1/AD$ and so on, take any pt. S and on any $|^s$ through S, take $SL = AD$, $SM = BE$, $SN = CF$. Let the $\odot$ LMN cut SL, SM, SN again at L', M', N'. Then $SL \,.\, SL' = SM \,.\, SM' = SN \,.\, SN'$, $\therefore$ $SL' \propto 1/SL \propto 1/AD$ and so on. Hence $BC : CA : AB :: SL' : SM' : SN'$. Construct the Δ $A'B'C'$ with sides SL', SM', SN', and let the altitudes be $A'D'$, $B'E'$, $C'F'$. Then $BC : CA : AB :: SL' : SM' : SN' :: B'C' : C'A' : A'B'$. Hence the Δ^s ABC and $A'B'C'$ are similar. Hence AB is given by $AB : AD :: A'B' : A'D'$; so BC, CA.

Ex. 3. Take the square $P'Q'M'N'$ and bisect $M'N'$ at L'. Then by symmetry the $\odot$ $P'L'Q'$ touches $M'N'$ at L'. Let MN touch at L; and construct the figure $NLMQP$ similar to the figure $N'L'M'Q'P'$. Then $PQMN$ is the required square. To perform the construction, let O and O' be the centres of the $\odot^s$; then $LM : LO :: L'M' : L'O'$ gives M; so N. Then $\perp^s$ to MN at M and N cut the $\odot$ at Q and P.

Ex. 4. Let $A'B'C'$ be of the required shape. On the same side of $B'C'$, $C'A'$ describe arcs containing the given angles BSC and CSA meeting again at S'. Now describe a figure $ABCS$ similar to $A'B'C'S'$ with side CA through D. To do this, draw DA through D, making the angle $DAS = C'A'S'$. Then DA cuts SC at C, and B is given by $\angle ACB = A'C'B'$.

Page 166. **§ 12.** **Ex. 1.** Let BC, ED meet at S. On CD, and towards A, describe the square $CDPQ$. Let SQ cut AB at L. With S as centre and Q, L as corresponding pts., describe a figure homothetic to $CDPQ$. The new figure is a square of which L (on AB) is the vertex corresponding to Q; hence by symmetry the pt. R corresponding

to P is on AE. Also the vertex M corresponding to C is on SC, i. e. BC; so N is on DE.

Ex. 2. As in the text, describe the square $CFPQ$. Then with A as centre and Q, N as corresponding pts., describe a figure homothetic to $CFPQ$. This is a square $RLMN$ with vertices R, L, M, N on AC, AB, AB, BC as required.

Page 166. § 13. Ex. 1. We want to prove that area $BCHE = \frac{1}{2}$ area $BCDA$. Now area $BCHE = BCE + CEH$ $= BCE + CEG$ [by $\|^s$ CE, HG] $= BCG = \frac{1}{2} BCF$ (since $BG = GF$) $= \frac{1}{2}(BCA + ACF) = \frac{1}{2}(BCA + ACD)$ [by $\|^s$ AC, DF] $= \frac{1}{2}(ABCD)$.

Ex. 2. Trisect AB at M, N. If P lies between M and N, draw MQ to AC and NR to BC, $\|$ PC. Then area $AQP = AQM + QMP = AQM + QMC$ (by $\|^s$ QM, PC) $= AMC = \frac{1}{3} ABC$ (since $AM = MN = NB$); so $BRP = \frac{1}{3} ABC$. Hence $QPRC = \frac{1}{3} ABC$. If P lies between N and B, take R on AC. Then area $PQR = PQA$ since

$$AQ : QR :: AM : MN.$$

Ex. 3. Take the fixed pt. A on l. Take the variable pt. P on m. On AP describe an equilateral Δ APQ; then by § 2, the locus of Q is a $|$ r, say. But C lies on n. Hence C is the int^n of n and r.

Ex. 4. On any line OA, take B, C such that $OA : OB$ is the given ratio and $OA \,.\, OC$ the given product. Take any $\odot$ c through A, C; and with O as centre and $OA : OB$ as ratio, describe the $\odot$ c', homothetic with c. Let these $\odot$s cut at P; and let OP cut c again at Q. Then $OQ : OP$ $:: OA : OB$ and $OQ \,.\, OP = OA \,.\, OC$. Hence OQ and OP are the required lines.

Ex. 5. Take two fixed pts. A, B. Then, if $AP^2 - PB^2$ is given, the locus of P is (p. 34) a $|$, l. Also if $AP : PB$ is given, the locus of P is (p. 66) a $\odot$, c. Hence, if both are given, P must be at an int^n of l and c.

Ex. 6. Bisect AB at E, and draw $EP \,\|\, BD$ to meet BC at P. Then $AP : PQ :: AE : EB :: 1 : 1$.

Ex. 7. Let $XA \,.\, AY = a^2$ and $XB \,.\, BY = b^2$. Draw

PAQ, $RBS \perp AB$ and bisected by AB and of such lengths that $PA = a$, $RB = b$. Then P, Q, S, R by symmetry lie on a ⊙. Let this ⊙ cut AB at M, N. Then M is X and N is Y. For $MA \,.\, AN = PA^2 = a^2$ and $MB \,.\, BN = RB^2 = b^2$.

Ex. 8. Let P be the pt. on l and PQ the normal distance to the given ⊙, c. Then $AP = PQ$. Hence the ⊙ with centre P and radius PA will touch c at Q. Hence we have to draw a ⊙ with centre on l to pass through A and to touch c. Now see p. 133, Ex. 6.

Ex. 9. Let the $\perp^s$ to the required |, x, be BM and CN. Bisect BC at D and draw $DR \perp x$. Then $DR = \frac{1}{2}(BM + CN)$. For draw $BP \perp DR$ and $DQ \perp CN$. Then the Δ^s BPD and DQC are congruent; for $BD = DC$, $P = Q = 90°$ and $\angle PBD = QDC$ by $\|^s$. Hence $DP = CQ$. Hence $BM + CN = DR - DP + DR + CQ = 2DR$. Hence DR is known, $= r$, say. Then R is the pt. of contact of a tangent drawn from A to the ⊙ with centre D and radius r.

Ex. 10. Call the circumference of the ⊙, c. Then each side of the polygon of m sides will cut off an arc of length c/m; so for the other polygon. We want an arc c/mn. Let us suppose that this can be got by taking the difference between x of the c/m arcs and y of the c/n arcs; then $xc/m \backsim yc/n = c/mn$ or $xn \backsim ym = 1$. Now if m and n are interprime and p/q is the penultimate convergent to m/n, we have $pn \backsim qm = 1$. Hence $x = p$ and $y = q$.

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Ex. 1. This is simply § 1, part 2, with the const. zero. In fact area $PLM = OLM$; hence the locus is a |.

Ex. 2. Making the given construction, area $CPQ = CA'Q + PA'Q = CA'Q + AA'Q$ (by $\|^s$ $A'Q$, PA) $= AA'C = \frac{1}{2}ABC$.

Ex. 3. Let x, y be the sides. Then the area $= xy \sin A$ is given, $\therefore xy = a^2$, say. Also the perimeter $2x + 2y$ is

given, $\therefore\ x+y=b$, say. Hence x, y are the roots of $z^2-bz+a^2=0$. Now see § 5.

Ex. 4. Let x, y be the $|^s$; then we are given x^2+y^2 and xy; i.e. x^2+y^2 and x^2y^2. Hence by § 5 we can construct x^2 and y^2. Then x and y can be found.

Or thus directly. Let $x^2+y^2=a^2$. Take $LM=a$ and on LM as diameter describe a $\odot$. Also on LM as base describe a rectangle $LMNR$ whose area is equal to the given product xy, so that $xy=LM.MN$. Let NR cut the $\odot$ at P, Q. Then $LP^2+MP^2=LM^2=a^2$ and $LP.PM = 2 .\text{area } LPM = LMNR = LM.MN = xy$. Hence $x=LP$, $y=PM$; or $x=PM$, $y=LP$. Q gives the same values.

Ex. 5. Take any pt. P on the $|$, l, on which B must lie and construct the ΔAPQ of the required shape. Then by § 2 the locus of Q is a $|$, x, say. But C must lie on the given $|$, m, say. Hence C is the int^n of x and m.

Ex. 6. Suppose we want to inscribe PQR in ABC so that QR shall be in a given direction. Take $Q'R'$ in this direction, Q' on CA and R' on AB; and on $Q'R'$, and towards BC, draw $P'Q'R'$ of the proper shape. Let AP' cut BC at P. Now draw PQR homothetic with $P'Q'R'$, taking A as centre and P, P' as corresponding pts. Then R is on AR', i.e. on AB; so Q is on AC. Also QR is $\parallel Q'R'$, i.e. in the right direction. Hence PQR is the required Δ.

Ex. 7. Let the radii of the $\odot^s$ on which P, Q, R must lie be a, b, c and the common centre be O. Take $P'Q'R'$ of the required shape. Obtain by p. 66, § 3, the locus of a pt. X such that $Q'X : R'X :: b : c$ and the locus of a pt. Y such that $R'Y : P'Y :: c : a$. Let these $\odot^s$ meet again at O'. On the given $\odot\ c$ take any pt. R and describe on OR a figure similar to the figure $O'P'Q'R'$. Then $QO : RO :: Q'O' : R'O'$ (by similar figures) $:: b : c$ and $RO=c$, $\therefore\ QO=b$; so $PO=a$.

Ex. 8. We are given R, A and b/c. But $a=2R\sin A$, $\therefore\ BC$ is given. Now A is given; hence one locus of A is

a ⊙. Also $BA:CA$ is given; hence, by p. 66, another locus of A is a ⊙. Hence A is at an int^n of these ⊙^{s}.

Ex. 9. We are given BC, $BA-CA$ and $C-B$. Let BD (on BA) $= BA-CA$, $\therefore AD = AC$, $\therefore \angle ACD = ADC$. Now $C-BCD = ACD = ADC = B+BCD$, $\therefore 2.BCD = C-B$. Hence BD and $\angle BCD$ are known. Draw CE so that $\angle BCE = \frac{1}{2}(C-B)$; and with centre B and radius $BA-CA$ describe a ⊙ cutting CE at D.

Ex. 10. We are given BC, $BA+AC$ and that A lies on a given |, l, say. Produce BA to D until $BD = BA+AC$, $\therefore AD = AC$. Hence a ⊙ with A as centre and AC as radius will pass through C and touch the known ⊙, r, with B as centre and BD as radius. This ⊙ can be described as the ⊙ touching r and passing through C and its reflexion in l. Then A is the centre of this ⊙.

Ex. 11. We are given BC and the lengths of AH, HD. Bisect BC at A' and draw $A'O \perp BC$ and $= \frac{1}{2}AH$. Then O is known. Hence A is on the ⊙ with centre O and radius OB. But AD is given. Hence A is on a ‖ to BC at a distance AD from it. Hence A is an int^n of this ⊙ with this ‖.

Ex. 12. We are given the lengths of AA', AH and R. Take A' anywhere and BC in any direction. Then $OA' \perp BC$ and $= \frac{1}{2}AH$ is known, $\therefore O$ is known. With O as centre and R as radius describe a ⊙; this cuts the base in B and C. A also lies on this ⊙; and also on a ⊙ with centre A' and radius AA'. Hence A is known as an int^n of two ⊙^{s}.

Ex. 13. Let P, Q lie on AB, AC. Through O draw OD to AB ‖ AC. Then $AD:DP :: QO:OP$ and is known. Hence P is known.

Ex. 14. To construct P so that the $\perp^s$ PX, PY, PZ on BC, CA, AB may be in a given ratio, say, $PX:PY:PZ :: l:m:n$. Draw $BQ \perp AB$ (towards C) $= m$ and $CR \perp AC$ (towards B) $= n$. Let ‖^s QS to AB and RS to AC meet at S. Draw SU, $ST \perp AB$, AC. Then $SU:ST :: BQ:CR$

$:: m : n$. Hence (by p. 155, Ex. 7) P must lie on the | AS. So another | BV can be found on which P lies. These |s meet at P.

Ex. 15. We are given $x^2 + y^2 = a^2$ and $x : y :: b : c$. Take $LM = a$ and on LM as diameter describe a ⊙, c_1. Then if P is any pt. on this ⊙, $PL^2 + PM^2 = LM^2 = a^2$. Again if $QL : QM :: b : c$, Q lies on a ⊙, c_2, by p. 66. Hence taking R at an intn of c_1, c_2 we have $RL^2 + RM^2 = a^2$ and $RL : RM :: b : c$. Hence $x = RL$, $y = RM$.

Ex. 16. Let the ⊥s be BQ, CR. Draw $CX \perp BQ$. Then $BX = BQ - CR$ which is given. Hence CX is a tangent from C to the ⊙ with B as centre and radius equal to the given difference. Then QR is ∥ CX.

CHAPTER XIV

Page 170. Ex. 1. Since the plane is ∥ l, we can take a | l' in the plane, ∥ l. Then the shortest distance between l and x is $\perp$ l and x, i. e. $\perp$ l' and x, i. e. $\perp$ the plane; and is ∴ the $\perp$ from any pt. on l to the plane. Also this is const. since l is ∥ the plane. If however $x \parallel l$, the shortest distance is $\perp$ both x and l in their plane and ∴ varies with the position of x.

Ex. 2. Let the |s be AB, CD, EF. Take any pt. P on AB and let the plane PCD cut EF at Q. Then PQ, being in the plane PCD, cuts CD. Hence PQ meets each of the |s.

Page 171. Ex. 1. Let the spheres intersect in the ⊙, c. Let the plane cut c in the pts. A, B and the spheres in the ⊙s $c_1, c_2, \ldots$. Then $c_1, c_2, \ldots$ all pass through A, B.

Now let c diminish to a pt. ⊙. Then the spheres touch; and the theorem is still true.

Ex. 2. Let AB cut the plane at O. Let P be the pt. of contact of a sphere through A, B. Then OP touches the sphere; hence $OP^2 = OA \,.\, OB$ is const. Hence the locus of P is a ⊙.

Ex. 3. Let A, B be the given pts. and P the variable pt. Then since $PA : PB$ is given, the locus of P is a sphere. The required locus is the $\odot$ which is the section of this sphere by the given plane.

Ex. 4. Let p be the required plane touching the given spheres 1, 2, 3. Then since p touches 1 and 2, it passes through a c. of s. S of 1, 2. So p passes through a c. of s. S' of 2, 3. Hence p is a tangent plane through the | SS' to 1. Hence there are 8 solutions; for S, S' and the tangent plane can be chosen in two ways.

Page 173. Ex. 1. The plane bisecting AB $\perp^{ly}$ is the locus of pts. equidistant from A and B; hence O lies on it. So each of the planes passes through O.

Ex. 2. Suppose the $\odot^s$ c_1, c_2 to meet at A and B. In c_1 take any other pt. P and in c_2 any other pt. Q. Then the sphere is that through A, B, P, Q.

Ex. 3. Invert the $\odot$ PQR w. r. to O. Then we get the $\odot$ $P'Q'R'$ such that $OP \,.\, OP' = OQ \,.\, OQ' = OR \,.\, OR'$. Hence the sphere $PQRP'$ passes also through Q', R'; and $\therefore$ through both $\odot^s$.

Ex. 4. Let the pts. be P_1, P_2 on AB and Q_1, Q_2 on AD and R_1, R_2 on AC and S_1, S_2 on CD and T_1, T_2 on BD and U_1, U_2 on BC. Consider the sphere $P_1Q_1S_1T_1$. This sphere cuts ABD in the $\odot$ $P_1Q_1T_1$ which by p. 81, Ex. 2, passes also through P_2, Q_2, T_2. Hence the sphere passes also through P_2, Q_2, T_2. Then from the plane BCD and the pts. T_2, T_1, S_1, the sphere passes through S_2, U_1, U_2. Next from the plane ACD and the pts. Q_1, Q_2, S_1, the sphere passes through R_1, R_2. Hence the sphere passes through all the pts.

Ex. 5. We have already considered, in p. 123, § 3, all the cases except when A, B, C, D do not lie in a plane. In this case consider the sphere $ABCD$. Invert w. r. to a pt. O on this sphere. Then A, B, C, D invert into A', B', C', D' on the plane which is the inverse of the sphere. Also, as on p. 124, $AB \,.\, CD + AD \,.\, BC > AC \,.\, BD$ if $A'B' \,.\, C'D'$

$+A'D'.B'C' > A'C'.B'D'$, i. e. unless A', B', C', D' lie on a $\odot$ (or a |) in the order $A'B'C'D'$, i. e. unless A, B, C, D lie on the inverse of a $\odot$ (or |), i. e. on a $\odot$ (or a |) in the order $ABCD$.

Page 174. Ex. 1. Vol. $GBCD$: vol. $ABCD$ :: altitude of $GBCD$: altitude of $ABCD$ :: $GA' : AA'$ :: $1 : 4$. Hence $GBCD = \frac{1}{4} ABCD = GACD = \ldots$.

Ex. 2. Let $AB = CD = a$, $AC = BD = b$, $AD = BC = c$. Then each of the faces has sides of length a, b, c; hence the faces are congruent. Again let L, L', M, M', N, N' bisect AB, CD, AC, BD, AD, BC. Then $AL : LB :: AM : MC$, $\therefore$ $LM \parallel BC$; also $LM : BC :: AL : AB$, $\therefore$ $LM = \frac{1}{2} BC$. So $M'L' \parallel BC$ and $= \frac{1}{2} BC$. So LM' and ML' are $\parallel AD$ and $= \frac{1}{2} AD$. But $BC = AD$; hence $LM'L'M$ is a rhombus. Hence LL', MM' are $\perp$. So $NN' \perp LL'$ and MM' and $\therefore$ $\perp$ plane $LM'L'M$ and $\therefore$ $\perp LM$ and $L'M'$ and $\therefore$ $\perp BC$ and AD; i. e. NN' is $\perp$ to the edges AD and BC. So for LL', MM'.

Ex. 3. (i) We know that vol. $GABC$ = vol. $GBCD = \ldots$. Also area ABC = area $BCD = \ldots$ in this case. Hence the $\perp$s from G on the faces are equal. Hence G is the incentre. (ii) Let G' be the centroid of the faces. Let $3m$ be the mass of each face. Then we can replace the face ABC by m at A, m at B, m at C; and so on. Hence we can replace all the faces by $3m$ at A, B, C, D. Hence G' is the c. of m. of equal masses at A, B, C, D and $\therefore$ coincides with G.

Ex. 4. (i) Let a plane $\parallel$ the edges BC, AD cut the edges AB, CD, AC, BD at L, L', M, M'. Now the three planes ABC, BCD, LML' meet at the intn of the |s LM, $M'L'$, BC; i. e. BC meets LM where it meets the plane LML', i. e. at infinity. Hence $LM \parallel BC$. So $L'M' \parallel BC$; and ML', $LM' \parallel AD$. Hence $LM'L'M$ is a $\parallel$m. (ii) If $LML'M'$ is a $\parallel$m, $LM \parallel M'L'$. But the three planes ABC, BCD, LML' meet in a pt., viz. at the intn of the |s LM, $M'L'$, BC. But $LM \parallel L'M'$; hence the intn is at infinity, i. e. $LM \parallel BC$. So $LM' \parallel AD$. Hence the plane $LML'M' \parallel BC$ and AD.

Ex. 5. With the figure of Ex. 4, area $LML'M' = LM \,.\, LM' \sin MLM'$. Now $\angle MLM'$ is const., being the angle between the $\|^s$ BC and AD to LM and LM'. Hence area $\propto LM \,.\, LM'$. Again, $LM : AL :: BC : AB$ and $LM' : BL :: AD : AB$. Hence area $\propto AL \,.\, LB$. But $AL + LB$ is const. Hence the area is greatest when $AL = LB$. Also $AL : LB :: AM : MC$, $\therefore AM = MC$; and so on.

Ex. 6. Let the tetrahedron be $ABCD$. If A is not such that $AB = AC = AD$, then keeping B, C, D fixed, move A on the spherical cap (bounded by the plane BCD) on which A lies until $AB = AC = AD$, i. e. until A is at the summit A' of the cap. Then vol. $A'BCD >$ vol. $ABCD$. Hence unless all the edges are equal we can increase the volume. Hence the volume is greatest when all the edges are equal.

Page 176. Ex. 1. With the figure of p. 175, we have seen that the altitude AL of the tetrahedron meets the altitude BE of the face BCD; so it meets each altitude of BCD and hence passes through the orthocentre of BCD.

Ex. 2. Consider the face BCD. Its N.P.C. passes through E since BE is an altitude; and similarly through the corresponding pts. on DB, BC; and also passes through the centres of the sides. So for the other faces. Now see p. 178, Ex. 4.

Page 177. Ex. Let AA' pierce the plane e at C. From P (in the plane e) draw $PP'' \perp | CP'$. Then $AC \perp$ plane e and $\therefore \perp PP''$. Hence $PP'' \perp CP''$ and AC and $\therefore \perp$ plane ACP'' and $\therefore \perp AP''$. $\therefore AP > AP''$; so $BP > BP''$, $\therefore AP + BP > AP'' + BP''$. But by p. 137, $AP'' + BP'' > AP' + BP'$, $\therefore AP + BP > AP' + BP'$. Hence $AP' + BP'$ is the shortest path.

End of Chapter XIV

Ex. 1. Suppose we have to draw a $| \,\|$ to AB to meet CD and EF. On CD take any pt. P and draw $PQ \parallel AB$. Let the plane QPD cut EF at X and draw $XZ \parallel PQ$ (and $\therefore$ to

AB). Then XZ being $\parallel PQ$ is in the plane QPX, i.e. in the plane QPD. Hence XZ meets CD since this also lies in the plane QPD. Also XZ meets EF and is $\parallel AB$.

Ex. 2. With the figure of p. 174, Ex. 2, by congruent Δ^s, $\angle BAD + DAC + CAB = BCD + DBC + CDB = 180°$.

Ex. 3. (i) With the figure of p. 174, Ex. 2, let the masses of the edges a, b, c be $2ka$, $2kb$, $2kc$. Then we may replace AB by ka at A and ka at B; and so on. Hence we get $k(a+b+c)$ at A, B, C, D. Hence the c. of m. of the sides is that of equal masses at A, B, C, D and is $\therefore$ G. (ii) It is sufficient to prove that $GA = GB = GC = GD$. But G bisects LL' and $L'L \perp AB$. Hence $GL, LA = GL, LB$ and $L = L$ ($= 90°$), $\therefore$ $GA = GB$; and so on.

Ex. 4. With the figure of p. 174, Ex. 2, the plane which passes through CD and L passes through LL' and $\therefore$ through G which bisects LL'.

Ex. 5. With the figure of p. 174, Ex. 2,

$$AB^2 + AC^2 = 2(N'A^2 + N'B^2)$$
$$DB^2 + DC^2 = 2(N'D^2 + N'B^2)$$
$$\therefore\ AB^2 + AC^2 + DB^2 + DC^2 = 2(N'A^2 + N'D^2) + 4N'B^2$$
$$= 4(N'N^2 + NA^2) + BC^2 = 4N'N^2 + AD^2 + BC^2,$$

or briefly $\quad 4z^2 = a^2 + b^2 + b'^2 + a'^2 - c^2 - c'^2$

where $\quad z = NN'$, $AB = a$, $CD = a'$, and so on.

$$\therefore\ 4z^2 = a^2 + a'^2 + b^2 + b'^2 - c^2 - c'^2,$$

so $\quad 4x^2 = b^2 + b'^2 + c^2 + c'^2 - a^2 - a'^2$

and $\quad 4y^2 = c^2 + c'^2 + a^2 + a'^2 - b^2 - b'^2,$

$$\therefore\ 4(x^2 + y^2 + z^2) = a^2 + a'^2 + b^2 + b'^2 + c^2 + c'^2.$$

Ex. 6. Let the two tetrahedrons be $ABCD$, $A'B'C'D'$. Then we are given that AA', BB', CC', DD' meet, at S, say. Consider the three planes ABC, $A'B'C'$, SAB. The three int^{ns} concur. Let the planes ABC, $A'B'C'$ meet in the $|\ l$; then AB and $A'B'$ meet on l, at P, say; so BC, $B'C'$ and CA, $C'A'$ meet on l, at Q, R, say. So the

int^{ns} of DB, $D'B'$ and of BC, $B'C'$ and of CD, $C'D'$ (L, Q, M, say) are collinear; also the int^{ns} of DC, $D'C'$ and of CA, $C'A'$ and of AD, $A'D'$ (M, R, N, say). We have now proved that PQR, LQM, MRN are |^s. Now since the |^s PQR, LQM meet at Q, the five pts. P, Q, R, L, M lie in a plane. Also since N lies on the | MR, N lies in this plane. Hence the six int^{ns} of corresponding edges are coplanar.

Ex. 7. Let the sphere touch AB, AC, AD, CD, DB, BC at L, M, N, L', M', N'. Then $AL = AM = AN$ ($= a$, say), being tangents from A; and so on. Hence $AB + CD = AL + BL + CL' + DL' = a + b + c + d$. So for the rest.

Ex. 8. A plane which touches two spheres must pass through one, S, of the c^s of s. If it also passes through the given pt. A, it must be one of the tangent planes from the | SA to either sphere.

Ex. 9. Take three pts. B, C, D on the ⊙ and draw a sphere through B, C, D and the given pt. A.

Ex. 10. This is the limit of p. 173, Ex. 2, when the two pts. coincide.

Ex. 11. Take any pt. P on the ⊙, and let OP cut the sphere again at P'; then $OP \cdot OP'$ is const. Hence the locus of P' is an inverse of the locus of P w. r. to O; i.e. is a ⊙.

Ex. 12. With the figure of p. 174, Ex. 4, we have $LM \parallel BC$ and $LM' \parallel AD$; hence if $LML'M'$ is a square so that $LM \perp LM'$, then $BC \perp AD$.

Again if $LML'M'$ is a square, $LM = LM'$. But $LM : BC :: AL : AB$ and $LM' : AD :: BL : AB$; hence $BC \cdot AL = AD \cdot BL$. Hence to get L we divide AB so that $AL : BL :: AD : BC$. Then draw $LM \parallel BC$, $ML' \parallel AD$ and $L'M' \parallel BC$.

MISCELLANEOUS EXAMPLES

PART I

Ex. 1. Let BX cut QP at C and BZ cut QR at A. It is sufficient to prove that AC passes through Y. Now the triangles ABC, $A'B'C'$ are copolar since AA', BB', CC' meet at Q. Hence they are coaxal, i.e. $(BC; B'C')$, $(CA; C'A')$, $(AB; A'B')$ are collinear, i.e. X, $(CA; C'A')$, Z are collinear. But XZ cuts $C'A'$ at Y. Hence CA passes through Y.

Ex. 2. Since O bisects PQ, the locus of P is the reflexion of l in O, i.e. is a $|$, l'. The required pts. are the intns of l' with c.

Ex. 3. If PX be the $\perp$, we have to prove that $PX.PA = PM.PN$, i.e. that $PX/PM = PN/PA$, i.e. that $\sin PMX = \sin PAN$, i.e. that $PMX = PAN$. And this is true since $PANM$ is cyclic.

Ex. 4. The $\odot$ on II_1 as diameter cuts BC at B and C. Then $\angle ABC = 2IBC$ and $\angle ACB = 2ICB$ gives A.

Ex. 5. Since the two $\odot^s$ on OA are equal, the angles ABO and ACO are equal ($= \alpha$, say). So $BAO = BCO = \beta$ and $CAO = CBO = \gamma$. Let AO cut BC at D. Then $\angle ADB = \alpha + \beta + \gamma = ADC$, $\therefore AD \perp BC$; and so on.

Ex. 6. Let A' bisect BC. Then OA' and $O_1A' \perp BC$, $\therefore OO_1 \perp BC$, i.e. $\parallel AH$, i.e. $\perp O_2O_3$; so $OO_2 \perp O_1O_3$, $\therefore O$ is the o. c. of $O_1O_2O_3$.

Ex. 7. $BX^2 - CX^2 = BP^2 - CP^2 = BQ^2 + QP^2 - PR^2 - CR^2$,

$$\therefore \quad BX^2 + CY^2 + AZ^2 - CX^2 - AY^2 - BZ^2$$
$$= BQ^2 + QP^2 - PR^2 - CR^2 + CR^2 + RQ^2 - QP^2 - AP^2$$
$$+ AP^2 + PR^2 - RQ^2 - BQ^2 = 0.$$

Ex. 8. D, E, F are collinear,

$$\therefore BD.CE.AF = -DC.EA.FB \quad \text{(i)}$$

So
$$BD'.CE'.AF' = -D'C.E'A.F'B \quad \text{(ii)}$$

Similarly if EF' cuts BC at D'', FD' cuts CA at E'', DE' cuts AB at F'', we have

$$BD''.CE.AF' = -D''C.EA.F'B \quad \text{(iii)}$$
$$BD'.CE''.AF = -D'C.E''A.FB \quad \text{(iv)}$$
$$BD.CE'.AF'' = -DC.E'A.F''B \quad \text{(v)}$$

Divide the product of (iii), (iv), and (v) by the product of (i) and (ii). Then

$$BD''.CE''.AF'' = -D''C.E''A.F''B.$$

Hence D'', E'', F'' are collinear.

Ex. 9. Let AC', DD' cut $A'C$ at X, Y. Let $A'D'$, CD meet at E. Then $AC'X$ gives

$$BA.A'X.CC' = -AA'.XC.C'B.$$

So $$A'D'.ED.CY = -D'E.DC.YA'.$$

i.e. $$BC'.A'A.CY = -C'C.AB.YA'.$$

Hence multiplying,

$$A'X.CY = XC.YA', \text{ i.e. } A'X/XC = A'Y/YC.$$

Hence X and Y coincide.

Ex. 10. The Δ^s ABC, $I_1I_2I_3$ are copolar (centre I); hence they are coaxal.

Ex. 11. Let AA', BB', ... meet at O. Then $\sin A'AB/\sin ABB' = OB/OA$; and so on. Now multiply.

Ex. 12.

$$AD'/D'B = OA.OD'\sin AOD'/(OD'.OB\sin D'OB)$$
$$= (OA/OB)\times(\sin AOD'/\sin D'OB);$$

and so on. Now multiply; and notice that $AOD' = A'OD$, $D'OB = DOB'$, and so on.

Ex. 13. By $\|^s$ BH, $A'H_1$, we have $HH_1 : H_1D : BA' : A'D$. By $\|^s$ CH, $A'H_2$, we have $HH_2 : H_2D :: CA' : A'D$. Hence $HH_1 : DH_1 :: HH_2 : H_2D$.

Ex. 14. Take the centre O of PQ as origin. Then $OP^2 = OQ^2 = OA.OB = OC.OD$, i.e. $p^2 = q^2 = ab = cd$. Hence $AC.AD.BP^2 = BC.BD.AP^2$ if

$$(c-a)(d-a)(p-b)^2 = (c-b)(d-b)(p-a)^2,$$

i.e. if $$(c-a)(p^2/c-a)(p-p^2/a)^2$$
$$= (c-p^2/a)(p^2/c-p^2/a)(p-a)^2,$$

i.e. if $$(c-a)(p^2-ac)(ap-p^2)^2$$
$$= (ac-p^2)(p^2a-p^2c)(p-a)^2,$$

i.e. if $(a-c)p^2(a-p)^2 = p^2(a-c)(a-p)^2$.

Ex. 15. Let QR cut AO at N and the $\perp$ bisector of QAR at T. Then (TN, QR) is h^c, $\therefore$ the polar of N passes through T. Also it is $\perp$ ON. Hence it is TA. Hence A, N are inverse pts. w. r. to the $\odot$, $\therefore$ N is a fixed pt. If, however, $QR \perp AO$, then T is a fixed pt.

Ex. 16. Let the ⊙s be a and b, O being on a. Let a | through O cut a again at P and b at Q, R. Then if (OP, QR) is h^c, P lies on the polar, p, of O w. r. to b. Hence P is one of the int^{ns} of p and a.

Ex. 17. Let AQ cut the ⊙ again at D' and PR at X. Now Q is the pole of PR ∴ (QX, AD') is h^c, ∴ $P(QX, AD')$ is h^c. But $P(QX, AD)$ is h^c by hyp. Hence PD and PD' coincide. Hence D' is at D (or C), i.e. AD (or AC) passes through Q; and so on.

Ex. 18. Let the centres of the polar ⊙s be U, U_1, U_2, U_3 and radii s, s_1, s_2, s_3. Then U, U_1, U_2, U_3 are at H, A, B, C. Also $s^2 = HA.HD$, $s_1^2 = AH.AD$, $s_2^2 = BH.BE$, $s_3^2 = CH.CF$. Now $UU_1^2 = s^2 + s_1^2$ if $AH^2 = HA.HD + AH.AD$, i.e. if $AH = AD + DH$; which is true. Hence the ⊙s u, u_1 are ⊥. Again, $U_2U_3^2 = s_2^2 + s_3^2$ if $BC^2 = BH.BE + CH.CF$, i.e. if $BC^2 = BD.BC + CD.CB$, i.e. if $BC = BD + DC$; which is true. So for the rest.

Ex. 19. Let EG cut DF at H. Then $\angle H = HDE + HED = TAG + UAG$ (if AT, AU touch b, c at A) $= TAU = 90°$.

Ex. 20. The three radical axes concur. Hence if AB, CD meet at O, OP is the common tangent at P. Also $OP^2 = OA.OB$ is const. Hence the locus of P is a ⊙ with centre O and radius $\sqrt{OA.OB}$.

Ex. 21. As on p. 136, the ⊙s QAR, RBP, PCQ meet at a pt., O, say. Then $A'B' \perp OR$ and $A'C' \perp OQ$. Hence $A' = 180° - QOR = A$; so $B' = B$, $C' = C$.

Ex. 22. Since $\angle APX = AP'X = 90°$, X is on the ⊙ APP' and AX is a diameter. Hence $ABX = 90°$. Hence the locus of X is the ⊥ BX to AB at B.

Ex. 23. Let the given ⊙ cut a fixed ⊙ through A, B in C, D and the required ⊙ through A, B in P, Q. Then if AB, CD cut at O, PQ passes through O. Hence we have to draw through the fixed pt. O the chord PQ of the given ⊙ so as to be of given length. Let E, e be the centre and radius of the given ⊙ and ER the ⊥ from E on PQ. Then

$ER^2 = e^2 - PR^2$ is known since $PR = \frac{1}{2}PQ$. Hence ER is known. Then PRQ is a tangent from O to a ⊙ with centre E and radius ER.

Ex. 24. Let the ⊙s intersect at A, B, and let XY pass through A. On XY take the pt. Z which divides XY in the given ratio. Let the ⊙ ABZ cut any other position $X'Y'$ of XY at Z'. Then, by p. 96, $X'Z' : Z'Y' :: XZ : ZY$. Hence the locus of Z' is the ⊙ ABZ.

Ex. 25. We know (p. 21) that I is the o. c. of the $\Delta I_1I_2I_3$. Hence the ⊙ ABC is the N. P. C. of the $\Delta I_1I_2I_3$ and hence bisects II_1, &c.

Ex. 26. Let the N. P. C. cut AH at X. Then $AH = 2 . AX$. Hence the locus of H is homothetic with the locus of X, i. e. is a ⊙ of twice the linear size of the N. P. C.

Ex. 27. Let HA' cut the circum⊙ at P'. Now H is the external c. of s. of the circum⊙ and N. P. C. and A, P' are the pts. on the circum⊙ corresponding to the pts. X, A' (p. 24) on the N. P. C. Hence AP' is a diameter of the circum⊙, since XA' is a diameter of the N. P. C. Hence P' is P. Hence $A'P$ passes through H; so $B'Q$, $C'R$.

Ex. 28. Let the variable ⊙ x touch the fixed ⊙ a and let the tangent OT to x from the fixed pt. O be const. Then x is ⊥ to the fixed ⊙ b with centre O and radius OT. Now see p. 114, Ex. 1.

Ex. 29. Call the ⊙s x, y, z. Invert w. r. to A. Then x inverts into a | x' passing through B', C', L' and ⊥ $D'AL'$; and so on. Hence A is the o. c. of the $\Delta B'C'D'$. Hence the ⊙ LMN inverts into the ⊙ $L'M'N'$, i. e. into the N. P. C. of $B'C'D'$. Also X is the other intn of the ⊙s ABC, LMN. Hence its inverse X' is the other intn of $B'C'$ with the N. P. C., i. e. X' is the centre of $B'C'$; so for Y', Z'. The ⊙s ADX, ABY, ACZ meet again if $D'X'$, $B'Y'$, $C'Z'$ concur, i. e. if the medians concur.

Ex. 30. Invert the four ⊙s c_1, c_2, c_3, c_4 into the concentric ⊙s a_1, a_2, a_3, a_4 with centre O and radii r_1, r_2, r_3, r_4. If Q, Q_1, Q_2, Q_3 are the inverses of P, P_1, P_2, P_3, then Q_1 is the inverse of Q w. r. to a_1, &c. Hence Q_1, Q_2, Q_3 lie on

OQ and we have $OQ \cdot OQ_1 = r_1^2$, $OQ_1 \cdot OQ_2 = r_2^2$, $OQ_2 \cdot OQ_3 = r_3^2$, $OQ_3 \cdot OQ = r_4^2$. And this is possible if $OQ/OQ_2 = r_1^2/r_2^2 = r_4^2/r_3^2$. Hence r_4 is given by $r_4 = r_1 r_3/r_2$. Hence a_4 is known. Inverting back, we get c_4.

Ex. 31. Let the given figure f be generated by the pt. P. It is sufficient to prove the theorem for P. Let P_1 be the inverse of P w. r. to the $\odot$ a and P_2 of P_1 w. r. to the $\perp$ $\odot$ b. Also let P_3 be the inverse of P w. r. to b and P_4 be the inverse of P_3 w. r. to a. We want to prove that P_4 coincides with P_2. Invert the $\odot^s$ into $\perp$ $|^s$ x, y. Then if P inverts into Q and so on, Q_1 is the reflexion of Q in x and Q_2 of Q_1 in y. Also Q_3 is the reflexion Q in y and Q_4 of Q_3 in x. Hence Q_4 and Q_2 coincide by inspection. Hence P_4 and P_2 coincide.

Ex. 32. Let OP, OQ cut c' again at P'', Q''. Then P'', Q'' are homothetic pts. of P, Q w. r. to O. Hence $P''Q''$, PQ are homothetic chords. Again by the quadrangle construction for the polar p' of O w. r. to c', $P''Q''$, $P'Q'$ meet on p', i.e. $P''Q''$ passes through R'. Also p', p are homothetic $|^s$. Hence R' (the int^n of $P''Q''$, p') is homothetic with R (the int^n of PQ, p). Hence RR' passes through O.

Ex. 33. Let the $\odot$ x touch the $\odot^s$ a, b of which L, M are the limiting pts. Invert w. r. to M. Then a, b become concentric $\odot^s$ a', b' and x' becomes a $\odot$ touching a', b' in a given manner. Hence the locus of the inverse of L' (which is now the centre of a', b') w. r. to x' is got by rotation about L' and is $\therefore$ a $\odot$ concentric with a', b'. Hence the inverse of L describes a $\odot$ of the system.

Ex. 34. Invert the coaxal $\odot^s$ into concentric $\odot^s$. Then evidently the locus of the inverses of A w. r. to these $\odot^s$ with centre O is the $|$ OA. Hence in the given figure the locus is a $\odot$.

Ex. 35. Consider A, B, C as pt. $\odot^s$ c_1, c_2, c_3 and call the other $\odot$, c_4. Then since the $\odot$ ABC touches c_1, c_2, c_3, c_4, we have, by p. 130, $12 . 34 \pm 14 . 23 \pm 13 . 24 = 0$. But 14 is the tangent from A to c_4; i.e. $14 = t_1$, and so on. Also $12 = AB$, and so on. Hence $AB . t_3 \pm t_1 . BC \pm CA . t_2 = 0$.

Ex. 36. Since $M = L = 90°$, O_3 bisects PC; so O_1 bisects PA and O_2 bisects PB. Hence $\Delta\ O_1O_2O_3$ is half $\Delta\ ABC$ (linearly).

Ex. 37. Since BA, AD and $\angle BAD$ are known, BD is known. Hence quad. $ABCD$ is greatest when $\Delta\ DBC$ is greatest, i.e. when $\angle DBC = 90°$.

Ex. 38. Let the given perimeter be $APQ \ldots B$, AB being the given base. On AB and on the same side of AB as $APQ \ldots B$, describe the $\odot^r$ arc $AP'Q' \ldots B$ having the given perimeter; and let the arc ALB complete the $\odot$. Then the two figures $APQ \ldots BLA$ and $AP'Q' \ldots BLA$ have the same perimeter; hence area $AP'Q' \ldots BLA > APQ \ldots BLA$ (by p. 139), $\therefore$ area $AP'Q' \ldots B > APQ \ldots B$.

Ex. 39. Let P' be a consecutive pt. to P. Then $AP : PB :: AP' : P'B$. Hence if E, F divide AB so that $AE : EB :: AP : PB :: AF : BF$, the $\odot\ c$ on EF as diameter passes through P and P', i.e. ult^{ly} touches the given $\odot$ at P. But (AB, EF) is h^c, $\therefore$ the $\odot\ ABP$ is $\perp\ c$ and $\therefore$ to the given $\odot$ since this touches c at P.

Ex. 40. Let $CR = x$, $PR = y$, and $CP = a$, $\therefore\ x^2 + y^2 = a^2$. Also $(x+y)^2 = 2(x^2+y^2) - (x-y)^2 = 2a^2 - (x-y)^2$ which is greatest when $x = y$. Hence $PR = RC$, $\therefore\ \angle CPQ = 45°$.

Ex. 41. Let PQ be the chord of the outer $\odot$ (centre O) which touches the inner $\odot$ (centre I) at X. From O draw $OY \perp PQ$. Then since $PQ = 2 . PY$ and $PY^2 + YO^2 = OP^2$, PQ is greatest and least when OY is least and greatest. Three cases arise. (i) O inside the inner $\odot$. Draw OZ $\perp\ IX$. Then $OY = IX \pm IZ$. Hence the least value of OY is $IX - IO$ (which is + since $IX > IO$) and the greatest value is $IX + IO$. Let IO cut the inner $\odot$ at A, B where A is nearer to O. Then OY is least and PQ greatest when X is at A; and OY is greatest and PQ least when X is at B. (ii) O outside the inner $\odot$. Then when PQ passes through O, PQ is greatest, being a diameter. Also OY has a max. and $\therefore$ PQ a min. when X is at A and at B; but PQ is least at B. (iii) O on the inner $\odot$. Then the tangent at O gives the greatest value of PQ and that at B the least.

Ex. 42. We have to inscribe in the $\Delta\, ABC$ a $\Delta\, PQR$ whose sides shall be $\parallel$ to those of the $\Delta\, P'Q'R'$. Through P', Q', R' draw $B'C'$, $C'A'$, $A'B' \parallel BC$, CA, AB. Then draw the figure $ABCPQR$ similar to the figure $A'B'C'P'Q'R'$. To do this, take P on BC so that $BP:PC :: B'P':P'C'$, and so on.

Ex. 43. We are given $\angle BAC$ and R, r. Take a $\odot$ with any centre I and radius r. Draw tangents AB, AC to it, making an angle A with one another. Let AB touch at X and suppose AB touches the $\odot$ escribed to BC at Y. Then $XY = AY - AX = s - (s-a) = a = 2R\sin A$ which is known, and $\therefore$ Y is known. Now take AZ on AC equal to AY. Then the escribed $\odot$ (viz. the $\odot$ touching AB, AC at Y, Z) is known. Now BC is either transverse tangent of the $\odot^s$.

Ex. 44. A particular case of p. 154, Ex. 2, if we notice that $(PAB)+(PAB)+(PAB)+(PCD)+(PCD)+(PCD)$ $+(PCD)$ is given.

Ex. 45. Let $PQ \parallel BC$ bisect ABC. Then $\Delta\, APQ : \Delta\, ABC :: 1:2$ (by hyp.) $:: AP^2 : AB^2$ (by similar Δ^s). Hence $AP = AB\,/\,\sqrt{2}$. To construct this, draw the square $ABDE$ and let AD, BE cut at F. Then $AF = FB$. Hence $AB^2 = AF^2 + FB^2 = 2AF^2$, $\therefore AP = AF$.

Ex. 46. Take the square $ABCD$. With centres A, B, C, D and radii equal to $\frac{1}{2}AB$, describe $\odot^s$. These touch one another. Let AC, BD cut at E. Produce EA, EB, EC, ED to cut the $\odot^s$ at L, M, N, R. Then $EL = EM = EN = ER$. Hence a $\odot$ with centre E and radius EL will touch the four $\odot^s$. Now take the given $\odot$ with centre E', and draw the figure $L'M'N'R'A'B'C'D'E'$ similar to the figure $LMNRABCDE$. To do this take $L'E'N' \parallel LEN$ and determine A' by $L'A' : A'E' :: LA : AE$, and so on.

Ex. 47. We are given that $\angle BAO = CAR$ and $\angle BAR = OQR$ and we have to prove that $AP\,.\,AQ = AR\,.\,AO$, i.e. that $AP/AO = AR/AQ$, i.e. that the Δ^s APO, ARQ are similar. Now $\angle PAO = RAQ$. Also $\angle AOP = OAQ + AQO = BAR + AQO = OQR + AQO = AQR$. Hence the Δ^s are similar.

Ex. 48. Let XY meet BC, BA at X, Y, be $\perp$ BC, and bisect ΔABC. Then $BX.BY = \frac{1}{2}BC.BA$. But $BX/BY = BD/BA$ if $AD \perp BC$. Hence (multiplying) $BX^2 = \frac{1}{2}BC.BD = BA'.BD$ if A' bisects BC. Hence X can be constructed.

Ex. 49. Bisect PQ at R. Then $AP.AQ = (AR-RP)(AR+RP) = AR^2 - RP^2$. Now $AP.AQ = AC.AD$ (if AC cut the $\odot$ again at D) since $Q = C = 90°$; also RP is given. Hence AR is known and $\therefore AP$ and $\therefore P$. In fact if $RP = c$ and $AC = a$, $AR = \sqrt{c^2 + 2a^2}$ which can be constructed by p. 160.

Ex. 50. We are given the angles B, C and the perimeter. Take any base $B'C'$ and make $\angle C'B'A' = B$ and $B'C'A' = C$. Then the Δ^s ABC, $A'B'C'$ are similar. Hence $BC : B'C' :: AB + BC + CA : A'B' + B'C' + C'A'$ gives the length BC.

Ex. 51. We are given a, A, and bc. Hence area $ABC = \frac{1}{2}bc \sin A$ is known. Now take $BC = a$. Then A is the int^n of two loci, viz. of the arc on BC containing the angle A and the $\parallel$ to BC at a distance p from it such that $pa = bc \sin A$. To construct p, take b' and c' at angle A such that $b'c'$ has the given value. Then describe on BC a triangle BCA'' of the same area as $A'B'C'$. Then the $\parallel$ to BC must be drawn through A''.

Ex. 52. We are given BC, A, and $c-b$. On BA take $BD = BA - AC$ so that $AD = AC$. Then BD is given. Also $\angle ADC = ACD$, $\therefore BDC = 180° - \left(90° - \frac{A}{2}\right) = 90° + \frac{A}{2}$ is known. Hence D lies on two $\odot^s$ and hence is known.

Ex. 53. Rotate B about l until the two planes coincide and A, B are on opposite sides of l. Then of course A, P, B must be collinear. Now rotate back again.

Ex. 54. Let the $|^s$ be l, m. Through any pts. on l, m draw l', m' $\parallel$ m, l. Then the planes lm' and $l'm$ are $\parallel$. Take a fixed position $X'Y'$ of XY. Through the centre Z' of $X'Y'$ draw the plane p $\parallel$ planes lm', $l'm$, cutting XY at Z.

Then $XZ : ZY :: X'Z' : Z'Y'$. Hence p bisects XY. Hence p is the locus required.

Ex. 55. Suppose, in the tetrahedron $ABCD$, that BAC is obtuse. Now $\angle BAC + BAD > DAC$. Also $BAC = BDC$, $BAD = BCD$, $DAC = DBC$, $\therefore \angle BDC + BCD > DBC$, $\therefore 180° - DBC > DBC \therefore DBC < 90°$.

Ex. 56. Let the $\perp$ from A on BCD be AG. Then, by symmetry, G is the centroid of the equilateral ΔBCD, and $AG \perp BG$. $\therefore 4BE^2 = 4BG^2 + 4GE^2$. But

$$BG = \tfrac{2}{3} . BL \text{ (if } BL \perp CD) = \tfrac{2}{3} BC\sqrt{3}/2 = BC/\sqrt{3}$$

$$\text{and} \qquad 4GE^2 = AG^2 = AB^2 - BG^2,$$

$$\therefore \quad 4BE^2 = 4BC^2/3 + AB^2 - BC^2/3 = AB^2 + BC^2$$
$$= 2BC^2 = 4CE^2 = 4DE^2 \text{ similarly.}$$

Hence $BE^2 + CE^2 = 2BE^2 = BC^2$. Hence $BE \perp CE$; so $CE \perp DE$ and $DE \perp BE$.

Ex. 57. Let L, M, N, R on OA, OB, OC, OD be such that $LMNR$ is a $\|^{m}$. Then the three planes OAB, OCD, LMN intersect where LM, RN intersect, i. e. at infinity. Hence LM, RN are $\|$ to the int^{n} x of the planes OAB, OCD; so MN, LR are $\|$ to the int^{n} y of the planes OBC, ODA. Hence take any pt. L on OA, draw $LM \| x$, $MN \| y$, $NR \| x$. Then $LR \| y$ because LR, MN, y (being the int^{ns} of OBC, OAD, $LMNR$) concur.

Ex. 58. Let the $\|^{m}$ be $PQRS$ (P on AB, Q on AC, R on CD, S on DB). Then $PQ/BC = AP/AB$, $\therefore PQ = AP . BC/AB$. Also $PS/AD = BP/AB$, $\therefore PS = BP . AD/AB$. Hence perimeter $= 2(PQ + PS) = 2(AP + BP) BC/AB$ (since $BC = AD$) $= 2AB . BC/AB = 2BC$; which is const. for sections $\| AD$, BC.

Ex. 59. PX, QY by symmetry are $\perp$ the diameter AO and $\therefore$ pass through the pt. I at infinity in a direction $\perp OA$. Again if QX, PY meet at R, the ΔIAR is self-conjugate w. r. to the $\odot$. Hence AR is the polar of I, $\therefore R$ lies on OA. Also A, R are conj. pts., $\therefore R$ is the pt. inverse to A and $\therefore$ fixed.

Ex. 60. This is the same as p. 73, Ex. 7. For the $\odot$ with centre P and radius t_1 is $\perp$ to the given $\odot$; so for Q.

Ex. 61. They form the coaxal system $\perp$ to the coaxal system determined by c and l. Hence they pass through the limiting pts. of the latter system.

Ex. 62. As on p. 17, $LB = LI = LC$; hence O_1 is at L, i.e. O_1 is on $\odot\, ABC$. So for O_2, O_3.

Ex. 63. We know (p. 21) that I is the orthocentre of $I_1I_2I_3$. Hence ABC is the N. P. C. of $I_1I_2I_3$. Hence K, the centre of I_2I_3, is on $\odot\, ABC$.

Ex. 64. $R_1 = AB/2 \sin ADB$ and $R_2 = AC/2 \sin ADC$. But $R_1 = R_2$ and $\sin ADB = \sin ADC \therefore AB = AC$.

Ex. 65. H is the external c. of s. of the N. P. C. and the circum$\odot$. Hence MN is homothetic to $A'D$ and $\therefore \parallel BC$. Hence the pt. is at infinity.

Ex. 66. Let the $\odot$ with centre O be called b. Invert the $\odot\, c$ (through $AQBO$) w. r. to $\odot\, b$. Then c inverts into the | through A, B. Hence Q inverts into P, i.e. P and Q are inverse w. r. to $\odot\, b$. Hence the polar of Q w. r. to b passes through P.

Ex. 67. Let TT' pass through the c. of s., S. Then the figures ATB, $A'T'B'$ are homothetic w. r. to S and are $\therefore$ similar.

Ex. 68. We are given $x - y = 2a$ and $xy = b^2$, $\therefore\ x + (-y) = 2a$ and $x(-y) = -b^2$. Hence x and $-y$ are the roots of the quadratic $z^2 - 2az - b^2 = 0$,
$\therefore z = a \pm \sqrt{a^2 + b^2}$, $\therefore x = a + \sqrt{a^2 + b^2}$, $y = \sqrt{a^2 + b^2} - a$.
Hence the construction. Take $PQ \perp PR$ of lengths a, b. Then $QR = \sqrt{a^2 + b^2}$. With Q as centre and a as radius describe a $\odot$ cutting RQ at X without, and Y within, RQ. Then $RX = RQ + QX = \sqrt{a^2 + b^2} + a = x$; so $RY = y$.

Ex. 69. We are given $x + y = 2a$, $x^2 + y^2 = 4b^2$,

$$\therefore\ 2xy = (x+y)^2 - (x^2 + y^2) = 4a^2 - 4b^2.$$

Hence x, y are the roots of the quadratic

$$z^2 - 2az + 2a^2 - 2b^2 = 0, \quad \therefore z = a \pm \sqrt{2b^2 - a^2},$$

$$\therefore\ x = a + \sqrt{2b^2 - a^2},\ y = a - \sqrt{2b^2 - a^2},$$

which can be constructed by p. 160.

Ex. 70. Since the base BC and the area are given, the locus of A is a $\parallel$ to BC, l say. Produce BA to D making $AD = AC$, $\therefore$ $BD = BA + AC$ which is known. Also a $\odot$ with centre at A and radius AC will touch the $\odot$ e with centre at B and radius BD. Hence A is the centre of a $\odot$ drawn to pass through C and to touch e and to pass through the reflexion of C in l.

Ex. 71. By p. 36, Ex. 8, $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4EF^2 = AC^2 + BD^2$ by hyp. Hence $EF = 0$. Hence E bisects both AC and BD. Hence AE, $EB = CE$, ED and $E = E$, $\therefore$ $\angle ABD = BDC$, $\therefore$ $AB \parallel CD$; so $BC \parallel AD$.

Ex. 72. In p. 37, § 9, put $m_1 = 1$, $m_2 = -1$, $m_3 = 2$, $m_4 = \ldots = 0$. Then

$$PA^2 - PB^2 + 2PC^2 = GA^2 - GB^2 + 2GC^2 + 2PG^2,$$

G being the centroid of 1, -1, 2 at A, B, C. Hence PG^2 is const. Hence the locus of P is a $\odot$ with centre at G.

Ex. 73. Required to place PQR with P on OA, Q on OB, R on OC. Place the ΔPQR in any position, say $P'Q'R'$. On $P'Q'$, $Q'R'$, and on the proper sides of them, describe arcs of $\odot$s containing the angles AOB, BOC. Let these arcs meet at O'. Then make $OP = O'P'$, $OQ = O'Q'$, $OR = O'R'$. Then since OP, $OQ = O'P'$, $O'Q'$ and $\angle POQ = P'O'Q'$, $\therefore$ $PQ = P'Q'$; so $QR = Q'R'$; and so $RP = R'P'$ since $POR = POQ + QOR = P'O'Q' + Q'O'R' = P'O'R'$.

Ex. 74. Area CED/area $BFD = (EM \,.\, CD)/(FN \,.\, BD) = (MC \,.\, AD)/(BN \,.\, AD)$ by similar triangles $= CM : BN$.

Ex. 75. $PA : PB :: AQ : BR :: a : b$. Now see p. 66.

Ex. 76. By p. 77, Ex. 1, find the locus of P such that $\angle APB = BPC$ and also the locus of P such that $\angle BPC = CPD$. Then P is an int$^{\text{n}}$ of these loci.

Ex. 77. With the figure of p. 24, the N. P. C. is the $\odot$ on $A'X$ as diameter. This always touches the $\odot$ with centre at A' and radius $A'X$; and this is a fixed $\odot$, because $A'X = R$ which is known, since $R = a/2 \sin A$.

Ex. 78. This is a particular case of p. 164, § 11.

Ex. 79. Required, in ABC, to inscribe PQR of given shape, so that P, Q, R shall be on BC, CA, AB and $PR \perp BC$. Take $P'Q'R'$ of the given shape and through P' draw $B'C' \perp P'R'$. Through Q', R' draw $Q'C'$, $R'B'$ making angles C, B with $B'C'$. Let $Q'C'$, $R'B'$ meet at A'. Now draw the figure $ABCPQR$ similar to the figure $A'B'C'P'Q'R'$. To do this, take P on BC so that $BP : PC :: B'P' : P'C'$, and so on.

Ex. 80. To describe the $\Delta\, OPQ$, so that O shall be a given pt. and P, Q be on the given $\odot^s$ a, b and $OP = OQ$ and $POQ = 90°$. Take P anywhere on a; and draw $OQ = OP$ and $\perp OP$. Then by p. 155, § 2, the locus of Q is a known $\odot$, c. Hence Q is an int^n of b, c.

Ex. 81. Take any position of P on the first $\odot$, and take R on PA produced such that $PA : AR$ is equal to the given ratio. Then the locus of R is a $\odot$ by p. 101, § 5. Then Q is an int^n of this $\odot$ with the second $\odot$.

Ex. 82. Drop the $\perp^s$ p and q on AB, CD. Then we are given that $p \,.\, AB = k \,.\, q \,.\, CD$ where k is a const., $\therefore p = c \,.\, q$ where c is a const. Now see the solution of Ex. 7, p. 155.

Ex. 83. Since $XE \,.\, XF = XB \,.\, XC$, X is on the r. a. of the N. P. C. and circum$\odot$; so Y, Z. Hence XYZ is the r. a. Hence $XYZ \perp NO$, i. e. $\perp GH$ (p. 102).

Ex. 84. Let the $|^s$ be AX, BY, CZ. Then $\angle BAX = 90° - AFE = 90° - C$. Hence AX passes through O; for (see p. 14) $\angle BAO = 90° - \frac{1}{2} AOB = 90° - ACB = 90° - C$. So BY, CZ.

Ex. 85. Invert w. r. to A. Then we want the locus of the int^n of the $|^s$ $P'R'$ and $Q'S'$, given that $P'AQ'$, $R'AS'$ are chords of a $\odot$. But $P'R'$, $Q'S'$ meet on the polar of A. Hence in the given figure the locus is a $\odot$ through A.

Ex. 86. Suppose we require P, Q, R to be on OA, OB, OC and QR, RP, PQ to pass through L, M, N where L, M, N are collinear. Take P anywhere on OA; let PN cut

OB at Q and let PM, QL meet at R. So construct $P'Q'R'$. Then since the Δ^s PQR, $P'Q'R'$ are coaxal, they are copolar, i. e. RR' passes through O. Hence taking R' as fixed and R as variable, if we attempt to construct the required triangle, R always lies on OR' instead of OC. Hence if OR' coincides with OC, the problem is indeterminate since we can take P anywhere on OA. Otherwise it is impossible.

Ex. 87. The centres of the $\odot^s$ ADX, BEY, CFZ bisect AX, BY, CZ since $D = E = F = 90°$. But X, Y, Z are collinear by Ex. 83. Hence the centres are collinear by p. 90, the sides of the quadrilateral being BC, CA, AB and XYZ.

Part II

Ex. 1. (i) Let N bisect CD and let O be the centre of the $\odot$. Then $AH_2 = 2 . ON = BH_1$. Hence $AH_2 =$ and $\parallel BH_1$, $\therefore H_2H_1 =$ and $\parallel AB$; so $H_3H_2 =$ and $\parallel BC$, and so on. Hence $ABCD$ and $H_1H_2H_3H_4$ are seen to be congruent on drawing the figure. (ii) Again $NG_1 = \frac{1}{3} NB$ and $NG_2 = \frac{1}{3} NA$, $\therefore G_2G_1 \parallel AB$ and $= \frac{1}{3} AB$. Hence $G_1G_2G_3G_4$ is similar to $ABCD$ and $\therefore$ to $H_1H_2H_3H_4$.

Ex. 2. Draw $OZ \perp PQ$. Then the $\odot^s$ on OP, OQ pass through Z; and so for OQ, OR and OR, OP. Hence the $|$ is the pedal $|$ of O w. r. to PQR.

Ex. 3. Draw $NQM \parallel BC$. It is sufficient to prove that $NQ = QM$. Now $PQZN$ is cyclic, since $\angle PQN = PZN$ $(= 90°)$ $\therefore \angle PNQ = PZQ = PZY = PAY$ (since $PZAY$ is cyclic) $= \frac{1}{2} A = PMQ$ similarly. Hence in the Δ^s PQN, PQM, we have $\angle PNQ = PMQ$, $PQN = PQM$ $(= 90°)$, $PQ = PQ$, $\therefore NQ = QM$.

Ex. 4. Let the four pts. be A, B, C, D. Let L, M, N, P, Q, R bisect AB, BC, CD, DA, AC, BD. We want to prove that the $\odot^s$ MNR, PQN, LPR, LQM meet in a pt. Let the $\odot^s$ MNR and LPR meet again at X. It is sufficient to prove that the $\odot^s$ PQN, LQM pass through X. $\odot$ PQN passes through X if $\angle PQN = PXN$. But $\angle PQN = ADC$ since $PQ \parallel DC$ and $QN \parallel AD$ and $\angle PXN = 360° - PXR$

$-RXN = PLR + RMN = ADB + BDC = ADC$. Hence $\odot\, PQN$, and so LQM, passes through X.

Ex. 5. The arc BAC is known since BC and angle A are given. Now the $\odot\, ABC$ is the N.P.C. of $I_1 I_2 I_3$ and hence cuts $I_2 I_3$ at the centre M' of $I_2 I_3$. Now L is on AI (p. 18) and $LAM' = 90°$ (i.e. LM' is a diameter); hence $M'(= M)$ bisects the arc BAC. Hence, if O' is the centre of the $\odot$ $I_1 I_2 I_3$, $II_1 = 2 . O'M$ (for $AH = 2 . OA'$ on p. 23) $\therefore MO' = \frac{1}{2} II_1 = LC$. But M is known and also LC. Hence the locus of O' is a $\odot$.

Ex. 6. With the figure of Ex. 1, the pedal $|$ a of A w. r. to BCD bisects AH_1; so the pedal $|$ of B w. r. to ACD bisects BH_2. But ABH_1H_2 is a $\|^m$. Hence the centres of AH_1, BH_2 coincide in X, say. Hence a and b pass through X. So b and c pass through the common centre of BH_2 and CH_3, i.e. c passes through X; and so d.

Ex. 7. With the figure of p. 26, Ex. 6, let O_1, O_2, O_3, O_4 be the centres of the $\odot$s ABC, AEF, CDE, BDF. Then PD is a common chord of $\odot$s 3, 4; hence $O_3 O_4 \perp PD$. So $O_4 O_1 \perp PB$. Hence $\angle O_3 O_4 O_1 = 180° - BPD$; so $\angle O_1 O_2 O_3 = APE$. Hence $O_3 O_4 O_1 + O_1 O_2 O_3 = 180°$, since $\angle BPD = BFD = APE$.

Ex. 8. Let OA, OB, OC, OA', OB', OC' meet $B'C'$, $C'A'$, $A'B'$, BC, CA, AB at X', Y', Z', X, Y, Z. Then we are given that X', Y', Z' are collinear and we have to prove that X, Y, Z are collinear. Since X', Y', Z' are collinear, by p. 41, Ex. 5, $\sin B'OX' . \sin C'OY' . \sin A'OZ' = \sin C'OX' . \sin A'OY' . \sin B'OZ'$. Now $B'OX' = YOA$, and so on. Hence $\sin AOY . \sin BOZ . \sin COX$

$$= \sin AOZ . \sin BOX . \sin COY.$$

Hence by the converse, X, Y, Z are collinear.

Ex. 9. Using the same converse as in Ex. 8, we have to prove that

$\sin BOD . \sin COE . \sin AOF = \sin COD . \sin AOE . \sin BOF$.

But $\sin BOD = \sin BOA' = \sin AOB' = \sin AOE$ and so on. Now substitute.

Ex. 10. By p. 47, Ex. 1, CQ is $\perp$ to the isogonal of AP w. r. to AB, AC. But $BA' : A'C :: BA : AQ$, $\therefore$ $CQ \parallel$ median AA', $\therefore$ AA' is $\perp$ isogonal of AP and $\therefore$ its isogonal AK is $\perp$ AP. Hence AP is $\perp$ symmedian AK.

Ex. 11. Let the proj^{ns} of K on BC, CB, AB be L, M, N. Then $KL : KM : KN :: a : b : c$. Hence area KLM : area $KMN :: ab \sin C : bc \sin A :: 1 : 1$. Hence $KLM = KMN = KNL$ similarly. Hence K is the centroid of LMN.

Ex. 12. Let Ω be the Brocard pt. of PQR for which $\angle RQ\Omega = QP\Omega = PR\Omega = \omega$. Then $\angle Q\Omega R = 180° - \omega - (R - \omega) = 180° - R = 180° - C$. Hence $Q\Omega RC$ is cyclic, and so on. Hence $\angle BA\Omega = PR\Omega = \omega = AC\Omega = CB\Omega$ similarly.

Ex. 13. In the figure on p. 60, we have to prove that βB, $\alpha A'$, $\gamma C'$ concur. In the second figure βB is a $\parallel$ to AA' through B, $\alpha A'$ is a $\parallel$ through A' to BB' and $\gamma C'$ is a $\parallel$ through γ to AB. Let the first two $\parallel^s$ meet at P. Then we have to prove that $\gamma P \parallel AB$. But γP, being a diagonal of the $\parallel^m$ $PB\gamma A'$, bisects $A'B$ and is $\therefore$ $\parallel$ AB since γ bisects $A'A$.

Ex. 14. Since $\angle TAC = ABC$, TA is $\parallel$ to antiparallels to BC. Draw the antiparallel $B'C'$ through K. Then K bisects $B'C'$. Hence $A(B'C', KI)$ is h^c, I being at infinity. Hence $A(BC, KT)$ is h^c since $AT \parallel B'C'$.

Ex. 15. Let EF meet BC at X. Then BC is a diagonal of the quadrilateral $FAEH$. Hence (BC, DX) is h^c. Hence P is X. Now see the solution of Ex. 83, p. 185.

Ex. 16. With the figure of p. 62, project UV to infinity. Then $ADCB$ becomes a $\parallel^m$ whose diagonals AC, BD meet at W. Also U is on AB and CD, and V on BC and AD, at infinity. Let L, M, N, R, X, Y be the six pts. in question on AD, BC, AB, CD, BD, AC. Then $LM \parallel AB$, $NR \parallel AD$ and X, Y are at infinity. Hence $AL : LD :: AN : NB$, $\therefore$ $LN \parallel BD$, i.e. L, N, X are collinear. So M, R, X and L, R, Y and N, M, Y are collinear. Hence L, N, X, Y, R, M are the six vertices of a quadrilateral.

Ex. 17. Take O as origin. Then $b = -a$ and $cd = a^2$. Also $OO' = x = \frac{1}{2}(OC + OD) = \frac{1}{2}(c + d)$. Hence

$$AC = c - a, BD = d - b = a^2/c + a = a(a + c)/c$$
$$CO = -c, CO' = x - c = \tfrac{1}{2}d - \tfrac{1}{2}c = \tfrac{1}{2}(a^2/c - c)$$
$$= \tfrac{1}{2}(a - c)(a + c)/c, BO = a$$
$$BO' = x + a = \tfrac{1}{2}c + \tfrac{1}{2}d + a = \tfrac{1}{2}c + \tfrac{1}{2}a^2/c + a = \tfrac{1}{2}(a + c)^2/c.$$

Now we have to prove that $AC . BO . BO' = BD . CO . CO'$

or $$(c - a)\,a\,.\,\tfrac{1}{2}(a + c)^2/c$$
$= [a(a + c)/c] . (-c) . \frac{1}{2}(a - c)(a + c)/c$, which is true.

Ex. 18. Let AP cut the ⊙ again at Q. Let the polar of A (which passes through B) cut AP at N. Let AP, CR cut at X. Then (AN, PQ) is h^c and $NBA = 90°$. Hence $\angle QBA = PBA = RCB$ (by reflexion). Hence $QB \parallel XC$, $\therefore AX : AQ :: AC : AB$ which is const. Hence locus of X is homothetic with locus of Q and is $\therefore$ a ⊙, x. Also A, C are homothetic to A, B. Hence A, C are inverse pts. w. r. to x, since A, B are inverse w. r. to the given ⊙. Again if AP' cuts CR at X', we prove as above that $AX' : AQ'$ $:: AC : AB$. Hence X' lies also on x.

Ex. 19. Let the tangents from A meet QP at L, M. Let the other tangent from L cut RQ at B'. Then the pole of LQ (viz. R) lies on LR; also LA, LB' are the tangents from L. Hence $L(AB', QR)$ is h^c. Hence B' is B, i.e. the other tangent from L passes through B; so the other tangent from M passes through B. Let AL, BM cut at X and AM, BL at Y and XY, AB at R' and LM, XY at P'. Then (AB, QR') is h^c by p. 60, $\therefore R'$ is R. Also RQP' is self-conjugate by p. 76. Hence P' is P.

Ex. 20. OA', HD are $\perp^s$ from the foci on the tangent BC. Hence A', D, and so on, lie on the auxiliary ⊙ which is $\therefore$ the N.P.C. whose radius is $\frac{1}{2}R$. Hence $2a = R$.

Ex. 21. To draw a ⊙ through A, B to cut CD h^{ly}. Suppose the int^{ns} with CD are X, Y. Then ⊙ $ABXY$ is $\perp$ ⊙ on CD as diameter. Now see p. 92, § 13, Ex. 1.

Ex. 22. On the given ⊙ a to find two pts. P, Q such that A, B, P, Q, and C, D, P, Q shall be concyclic. Let any ⊙

through A, B cut a in E, F; then AB, EF, PQ concur. Let any ⊙ through C, D cut a in G, H. Then CD, GH, PQ concur. Let AB, EF meet at U and CD, GH at V. Then UV cuts a at P, Q. For $UA \,.\, UB = UE \,.\, UF = UP \,.\, UQ$, $\therefore$ $ABPQ$ is cyclic; so $CDPQ$.

Ex. 23. Let (1, 2) denote the r. a. of i_1 and i_2, and so on. Then (0, 1) $\perp$ II_1 and (2, 3) $\perp$ I_2I_3. Hence (0, 1) $\perp$ (2, 3), and so on. Also (0, 1) (1, 2) (2, 0) concur, and so on.

Ex. 24. With S, one of the pts. in which OA cuts c, as centre, form a figure homothetic to c, taking O to correspond to A. Then ⊙ c', homothetic to c, touches c at S. Also the tangents to c' from O are homothetic to the tangents from A to c and are $\therefore$ $\parallel$ to these, i.e. are l, m.

Ex. 25. See the figure of p. 60. Let I, r be the centre and radius of the given ⊙ and O_1, O_2, O_3, O_4 and R_1, R_2, R_3, R_4 the centres and radii of the ⊙s ABC, $AB'C'$, $A'BC'$, $A'B'C$ which meet again at P. Let S be the external c. of s. of ABC, $AB'C'$. Then S is the centroid of R_2 at O_1 and $-R_1$ at O_2. Hence if X is any pt. we have, by p. 36, $R_2 \,.\, XO_1^2 - R_1 \,.\, XO_2^2 = R_2 \,.\, SO_1^2 - R_1 \,.\, SO_2^2 + (R_2 - R_1)\, SX^2$. Take X to be P, $\therefore$ $XO_1 = R_1$, $XO_2 = R_2$, $\therefore$ $R_2R_1^2 - R_1R_2^2 = R_2 \,.\, SO_1^2 - R_1 \,.\, SO_2^2 + (R_2 - R_1)\, SP^2$. Take X to be I, $\therefore$ $XO_1^2 = O_1I^2 = R_1^2 + 2R_1r$, $XO_2^2 = O_2I^2 = R_2^2 + 2R_2r$, $\therefore$ $R_2\ (R_1^2 + 2R_1r) - R_1\ (R_2^2 + 2R_2r) = R_2 \,.\, SO_1^2 - R_1 \,.\, SO_2^2 + (R_2 - R_1)\, SI^2$. Hence $SP = SI$. Hence S lies on the $\perp$ bisector of P, I. So other cases can be discussed, noticing that we take the internal c. of s. if the given circle is inscribed in one triangle and escribed to the other as in the case of O_1 and O_4.

Ex. 26. It is sufficient to show that (AL', BM', CN'), (AL' BM, CN), (AL, BM', CN), (AL, BM, CN') concur, L, M, N being the external cs of s. AL, BM, CN' concur if $BL/LC \,.\, CM/MA \,.\, AN'/N'B = 1$, i.e. if $-b/c \,.\, -c/a \,.\, a/b = 1$; which is true. So for the rest.

Ex. 27. Let I be the centre and r the radius of the fixed inner ⊙. Let AI cut the outer fixed ⊙ again at L. Let I', r' be the centre and radius of the inscribed ⊙. Then

$AI \,.\, IL$ is const. Also $AI' \,.\, I'L = 2Rr'$ (p. 17) and $AI : AI'$ $:: r : r'$. Hence $LI : LI' \propto 1/AI : r'/AI' \propto r$ is const. $= k$, say, $\therefore II' = IL \,.\, (k-1)/k$. But I is fixed and L moves on the outer $\odot$. Hence I' moves on a homothetic $\odot$.

Ex. 28. We know that the centres L, M, N of AA', BB', CC' are collinear. To prove that C is on the $\odot$ of s., we may show that $CL : CM :: AL : BM$, i.e. that $CL/LA = CM/MB$, i.e. that the Δ^s CLA, CMB are similar. Now (see fig. of p. 60) the Δ^s $CA'A$, $CB'B$ are similar; for $\angle CA'A = CB'B$ and $C = C$. Hence $\angle CAA' = CBB'$ and $CA : AA' :: CB : BB'$, $\therefore$ $CA : AL :: CB : BM$. Also $\angle CAL = CBM$. Hence Δ^s CLA, CMB are similar. Hence C is on the $\odot$ of s. So C' is on the $\odot$ of s. Hence the centre of the $\odot$ of s. is on the $\perp$ bisector of CC' and on LM and is $\therefore N$. Hence the $\odot$ on CC' as diameter is the $\odot$ of s.

Ex. 29. Let U, V be the centres of the $\odot^s$ on AB, AC as diameters. Let S, S' be the c^s of s. Then (SS', UV) is h^c. Hence the $\odot$ of s., viz. the $\odot$ on SS' as diameter, is $\perp \odot$ AUV and $\therefore \perp \odot$ ABC which is homothetic with $\odot$ AUV, A being the centre and $1:2$ being the ratio; for the homothetic $\odot^s$ touch at A, which is on the $\odot$ of s.

Ex. 30. Invert w. r. to C. Then a, b become $|^s$ a', b' cutting at D'; and X', C, Y' are collinear. The $\odot$ through $D, X \perp a$ becomes the $\odot$ on $D'X'$ as diameter; so $D'Y'$. These $\odot^s$ cut at P', the proj^n of D' on $X'Y'$. Hence the locus of P' is the $\odot$ on $D'C$ as diameter. Hence the locus of P is the $|$ through $D \perp CD$.

Ex. 31. Let X, Y, Z be the centres and AP, AQ, AR diameters of the $\odot^s$. Then since $AP = 2AX$, $AQ = 2AY$ $AR = 2AZ$, it is sufficient to prove that A, P, Q, R are concyclic. Invert w. r. to A. Then $AB'C'D'$ is cyclic. Also Q' the inverse of Q is the proj^n of A on $C'D'$, and so on. Hence, by the pedal theorem, P', Q', R' are collinear. Hence P, Q, R, and $\therefore X, Y, Z$, lie on a $\odot$ through A.

Ex. 32. Let the $\odot^s$ AOP, BOQ cut at R. Invert w. r. to O. Then $P'Q'$ is still a diameter. Also the $\odot^s$ AOP, BOQ become the $|^s$ $A'P'$, $B'Q'$ meeting at R'. We want to prove

that the locus of R' is a $\odot$ through A', B' which is $\perp$ $\odot$ $A'B'P'Q'$. Now $\angle R' = 180° - P' - Q' = 180° - A'P'B' - B'P'Q' - B'Q'P' = 90° - A'P'B'$ which is const. Hence the locus of R' is a $\odot$ through A', B'. Again $\angle OA'P' = OP'A' = A'B'R'$. Hence OA' touches the $\odot$ $A'B'R'$. Hence the $\odot^s$ are $\perp$.

Ex. 33. Invert w. r. to the $\odot$ itself. Then A inverts into the centre A' of LM and so on. Now $\angle LEM = 90°$. Hence $A'E = A'L = A'M$. Hence $A'O^2 + A'E^2 = A'O^2 + A'L^2 = OL^2 = B'O^2 + B'E^2 = \ldots$ similarly. Hence by p. 36, Ex. 1, the pts. A', B', ... lie on the same fixed $\odot$. Hence A, B, C, D lie on the same fixed $\odot$.

Ex. 34. Invert the $\odot$ w. r. to O, taking P' to correspond to P. Then the $\odot$ inverts into itself, Q into Q', Q' into Q and the int^n R of the $\odot^s$ OPQ, $OP'Q'$ into the int^n R' of the $|^s$ $P'Q'$, PQ. But the locus of R' is the polar of O, i. e. the $|$ through O', the inverse of O w. r. to the given $\odot$, $\perp$ OO'. Also the inverse of O' is C (p. 112, Ex. 1). Hence the locus of R is the $\odot$ on OC as diameter.

Ex. 35. Suppose we are projecting from the pt. O (on the given sphere e) on to the inverse plane e'. Let V be the vertex of the tangent cone along the contour of the given circle c on the sphere and P a pt. on c. With V as centre and VP as radius describe a sphere s. Then s is $\perp$ sphere e. Let OV cut e' at V'. Then the inverse sphere s' $\perp$ plane e'. Hence its centre is on e' and also on OV and $\therefore$ at V'. Hence the inverse $\odot$ c' (being the section of s' and e') also has its centre at V'.

Ex. 36. For brevity let $PA = a$, and so on; $AB = BC = \ldots = p$, $BE = AC = \ldots = q$. Then from $PABE$

$$pb = pe + qa \quad \ldots \ldots \quad \text{(i)}$$

so from $PADE$ $\quad pd = pa + qe \quad \ldots \ldots \quad$ (ii)

and from $PACE$ $\quad pc = qa + qe$

$$\text{i. e. } \quad 0 = -qa + pc - qe \quad \ldots \ldots \quad \text{(iii)}$$

Now add (i), (ii), (iii), $\therefore$ $pb + pd = pe + pa + pc$,

$$\therefore \; b + d = a + c + e.$$

Ex. 37. (i) Invert w. r. to the inner ⊙. Then A inverts into A', the centre of QR; and so on. Hence the ⊙ $A'B'C'$ (viz. the N. P. C. of PQR) is the inverse of the outer ⊙ and is ∴ fixed. But this touches the ⊙ inscribed in PQR. Hence the ⊙ inscribed in PQR touches a fixed ⊙. Also G', H' of PQR are the c^s^ of s. of the N. P. C. and circum⊙ of PQR, i. e. of fixed ⊙^s^ and are ∴ fixed. (ii) Invert w. r. to the outer ⊙. Then if the Δ formed by the tangents is XYZ, the inverse X' of X bisects BC and so on. Hence ⊙ XYZ inverts into the N. P. C. of ABC which touches the inner ⊙. Hence the ⊙ XYZ touches a fixed ⊙.

Ex. 38. By Ex. 23, R_1, R_2, R_3, R_4 form a triangle and its o. c. Also R_2R_3 and R_1R_4 cut at A', the centre of BC, and so on (by p. 80, Ex. 4). Hence the N. P. C. of ABC (viz. $A'B'C'$) is the N.P.C. of $R_1R_2R_3$ and ∴ touches the required ⊙^s^.

Ex. 39. Let $APQ \ldots B$ be the given figure on the given base AB and of given area. Let $AP'Q' \ldots B$ be the ⊙^r^ arc of the same area. Complete the ⊙ $AP'Q'BCA$. Then the ⊙ $AP'Q'BCA$ and the figure $APQBCA$ have the same area. Hence by p. 141, § 10, perimeter $A'P'Q'BCA < APQBCA$, i. e. $A'P'QB < APQB$.

Ex. 40. Let the ⊥^s^ from A, B on the tangent at P be AQ, BR. Draw the ⊥^s^ PX, PY, PZ to the tangents at A, B and to AB. Now, by symmetry, $AQ = PX$ and $BR = PY$. Hence $AQ.BR = PX.PY = PZ^2$ (by p. 27). Hence PZ must be greatest, i. e. P must be the extremity of the diameter ⊥ AB which is furthest from AB.

Ex. 41. Since AX, B_1C_1, C_2B_2 concur,

$$\sin BAX \,.\, \sin B_2B_1C_1 \,.\, \sin B_1C_2B_2,$$
$$= \sin XAC \,.\, \sin C_1B_1C_2 \,.\, \sin B_2C_2C_1.$$

Hence $\sin BAX/\sin XAC$

$$= (\sin C_1B_1C_2 \,.\, \sin B_2C_2C_1)/(\sin B_2B_1C_1 \,.\, \sin B_1C_2B_2),$$

and so on. Hence

$$(\sin BAX/\sin XAC).(\sin CBY/\sin YBA).(\sin ACZ/\sin ZCB)$$
$$= \frac{\sin C_1B_1C_2 \,.\, \sin B_2C_2C_1 \,.\, \sin A_1C_1A_2 \,.\, \sin C_2A_2A_1}{\sin B_2B_1C_1 \,.\, \sin B_1C_2B_2 \,.\, \sin C_2C_1A_1 \,.\, \sin C_1A_2C_2}$$
$$\frac{.\, \sin B_1A_1B_2 \,.\, \sin A_2B_2B_1}{.\, \sin A_2A_1B_1 \,.\, \sin A_1B_2A_2}.$$

And this is unity; for $C_1B_1C_2 = C_1A_2C_2$, and so on. Hence AX, BY, CZ concur.

Ex. 42. A_1B_1 bisects both AA' and AA'', hence $A'A'' \parallel A_1B_1$; so $B'B'' \parallel A_1B_1$. Draw $A'M$, $AN \parallel A_1B_1$ and $B'MN \perp A_1B_1$; and let AN, BB'' cut at R. Then $B'B'' = NR = NA - RA = MA'' - RA = A'A''$ if $RA = MA'' - A'A'' = MA'$. Now B, B'' and $\therefore$ B, B' are equidistant from A_1B_1; and similarly R, M; hence $BR = B'M$. Also $BA = B'A'$ and $R = M = 90°$. Hence $AR = A'M$. Hence we can get AB to $A'B'$ by the reflexion and translation stated; and AB will carry the figure with it.

Ex. 43. Take AC in any position. Then $\angle ACB$ gives CB in direction and $\angle CAD$ gives AD in direction. Hence we have to place a | BC of given length and direction between the given |s CB, DA, meeting at O, say. Take any | $B'D'$ in the right direction. Then $OB : OB' :: BD : B'D'$ gives B and $\therefore$ D.

Ex. 44. Reflexion gives congruent figures of different kinds. Hence an even number of reflexions gives congruent figures of the same kind. The common pt. of these figures is their c. of i. rotation. This gives P.

Ex. 45. Let the |s meet at O. Take $A'X' = A'Y'$ at an angle equal to the given value of XAY. On $A'X'$, $A'Y'$, and on the proper sides, describe arcs meeting at O' and containing the given angles AOX and AOY. Then draw the figure $OXAY$ similar to $O'X'A'Y'$. To do this, take $OX : OA :: O'X' : O'A'$ and $OY : OA :: O'Y' : O'A'$.

Ex. 46. Suppose $\Delta BCD > \Delta ABD$. To bisect $ABCD$ by a | through B, draw AE to $CD \parallel BD$ and bisect CE at F; then BF bisects $ABCD$. For $\Delta BFC = \frac{1}{2}BCE = \frac{1}{2}BCD + \frac{1}{2}BDE = \frac{1}{2}BCD + \frac{1}{2}BDA = \frac{1}{2}$. area $ABCD$.

Ex. 47. Suppose $AX \,.\, AY = a^2$ and $BX \,.\, BY = b^2$. Describe ⊙s with centres A, B and radii a, b; and take their limiting pts. L, M. Then L, M are inverse w. r. to the first ⊙, $\therefore$ $AL \,.\, AM = a^2$; so $BL \,.\, BM = b^2$. Hence X, Y are L, M.

Ex. 48. To inscribe the rhombus $D'E'B'A'$ so that the pts. D', E' shall be on AB, D' being given. With centre A and radius AB, draw a ⊙ cutting CD' at D. Complete the rhombus $BADE$; and with C as centre describe a figure homothetic to $BADE$, taking D' to correspond to D. This is the required rhombus. For D becomes D' and A', B' lie on CA, CB. Also $D'E' \parallel DE$, hence E' is on AB.

Ex. 49. Assume AD. Then B, C, A', L lie on the $\perp$ to AD at D, AL being the bisector. Hence the lengths of AA' and AL give A' and L. Again AO and AD are equally inclined to AL. Hence O is known as the intⁿ of AO and the $\perp$ to DA' at A'. Then $OB = OC = OA$ gives B and C.

Ex. 50. If the ⊙ʳ sections are not $\parallel$, let them meet in the (real or imaginary) pts. A, B. Through the fixed generator OCD of the cone (with vertex O) draw a plane section of the figure, cutting AB in X and the ⊙ˢ in C, Q and D, P. Then $CX \cdot XQ = AX \cdot XB = DX \cdot XP$. Hence $CDQP$ is cyclic. Hence $OP \cdot OQ = OC \cdot OD =$ const. and O, P, Q are collinear; hence the locus of Q is the inverse of the locus of P, i. e. one section is the inverse of the other.

Ex. 51. Since the faces are congruent, the circum⊙ˢ of the faces are equal. Hence (see p. 172) if A', B', C', D' are the circumcentres of the faces, OA', OB', OC', OD' are $\perp$ the faces and equal. Hence the inscribed sphere touches the faces at A', B', C', D'.

Ex. 52. The tangent plane at V cuts the sphere in a pt. ⊙; and hence the other ⊙ʳ sections are $\parallel$ to it.

Ex. 53. Let the successive pts. of reflexion be L, M, N, R. Then if $\angle ALE = \theta$, $BLM = \theta$, $\therefore BML = 90° - \theta = CMN$, $\therefore CNM = \theta = DNR$, $\therefore DRN = 90° - \theta = ARE$. Hence R, E, L are collinear and $LMNR$ is a $\parallel^m$ and the Δ^s LBM, NCM are similar. Hence $LB : BM :: NC : CM :: LB + NC : BM + CM :: LB + AL : BC :: AB : BC$; for the Δ^s NCM and ALR are congruent since $NM = LR$. Now since $LB : BM :: AB : BC$, LM is $\parallel AC$. So $RN \parallel AC$; and MN, $LR \parallel BD$.

Ex. 54. Bisect AB at P. Then since P bisects AB and A' bisects BC, $A'P \parallel AC$. Hence, relatively to AA', the locus of P is an arc of a $\odot$ containing an angle $180° - A$. Also $\Delta\ ABC = \frac{1}{2} AB \,.\, AC \sin A = 2 AP \,.\, PA' \sin P = 4\,\Delta\, APA'$. Draw $PM \perp AA'$. Then $\Delta\, APA'$ is greatest when PM is greatest, i. e. when P bisects the arc AA', i. e. when $AP = PA'$, i. e. when $AB = AC$. Hence $\Delta\, ABC$ is greatest when $AB = AC$.

Ex. 55. Take $CP'Q'$ consecutive to CPQ. Then CPQ is a critical position if

$$\begin{aligned} \text{area } PBQ + \text{area } PCD &= \text{area } P'BQ' + \text{area } P'CD, \\ \text{or} \quad PCD - P'CD &= P'BQ' - PBQ, \\ \text{or} \quad \text{area } CPP' &= \text{area } PP'Q'Q, \\ \text{or} \quad \text{area } CQQ' &= 2 \text{ area } CPP', \\ \text{or} \quad \text{ult}^{\text{ly}}\ CQ^2 &= 2\,CP^2, \ \therefore\ CQ = CP\sqrt{2}. \end{aligned}$$

Draw PX, $QY \perp CD$. Then $PX : QY :: CP : CQ :: 1 : \sqrt{2}$ gives PX, and $\therefore$ P, by drawing a | at distance PX from CD to cut DB at P. Also this critical position is unique, and $\therefore$ makes (area PBQ + area PCD) least, if the critical value is less than the extreme values. Now Q ranges from B to A; hence in both extreme positions (area PBQ + area PCD) is equal to half the area of the $\|^{\text{m}}$, say Δ. Also $(PBQ + PCD) = CQB - CPB + \Delta - CPB = \Delta + CQB - 2\,CPB < \Delta$ if $2\,CPB > CQB$, i. e. if $2\,CP > CQ$, i. e. if $2\,CP > CP\sqrt{2}$; which is true.

Ex. 56. Since (p. 137) the shape of QOR is const. and $QX : XR$ is const., the shape of OXQ is const. Hence (p. 155) since Q moves on a | and O is fixed, X moves on a |.

Ex. 57. By p. 150, if LPM is the tangent at P, $\angle LPA = MPB$. Hence if O is the centre of the $\odot$, $\angle\, OPA = OPB$. But since $OA \,.\, OA' = OP^2$, the Δ^{s} OAP, OPA' are similar,

$$\therefore\ \angle\, OPA = OA'P;\ \text{so}\ OPB = OB'P.$$

Hence $PA'B' = PB'A'$, $\therefore$ $PA' = PB'$. Greatest because an ellipse with foci A, B and touching the $\odot$ at P will be outside the $\odot$.

Ex. 58. By p. 22, the reflexion of $\odot\, BHC$ in BC is the $\odot\, ABC$. Hence O_1 is the reflexion O in BC. Hence $OO_1 \perp BC$

and is bisected by A'. Again $O_2O_3 \perp AH$ and $\therefore \perp OO_1$; and so on. Hence O is the o. c. of $O_1O_2O_3$. But A' bisects OO_1. Hence A' is on the N. P. C. of $O_1O_2O_3$; so B', C'. Hence the N. P. C.s coincide.

Ex. 59. By p. 17, if AI cuts $\odot$ ABC at L, $LB = LI = LC$. Hence L is the centre of the $\odot$ BIC. But L is equidistant from AB, AC. Hence, by symmetry, $AP = AC$ and $AQ = AB$. Hence since BC touches the in$\odot$, PQ also touches it.

Ex. 60. Let P, Q move on the $|^s$ l, m. Since angle POQ is given, we want $OP \,.\, OQ$ least. Rotate OQ and m about O until Q comes on PO at R, and let n be the new position of m. Then P and R are collinear with O and move on the $|^s$ l, n. Now $PO \,.\, OR$ is critical if $PO \,.\, OR = P'O \,.\, OR'$, i. e. when $PP'RR'$ is cyclic, i. e. ult^{ly} when P, R are the pts. of contact of a $\odot$ touching l, n. Hence POR is $\parallel$ to the external bisector of the angle between l, n. Also this gives a unique critical value separated by infinite values when POR is $\parallel$ to l or n. Hence it gives a minimum.

Ex. 61. $A'B'C'$, DEF are triangles inscribed in the N. P. C. Hence we have to prove that the pedals $N'L'$, NL of any pt. P on the $\odot$ are $\parallel$. Now referring to p. 26, Ex. 2, $N'L'C' = 90° - PB'A'$ and $NLF = 90° - PED$,

$$\therefore\; NLF - N'L'C' = PB'A' - PED.$$

But if $NL \parallel N'L'$, then $NLF - N'L'C'$ is equal to the angle between EF, $B'C'$; also $PB'A' - PED$ is equal to the sum of the angles between ED and $B'A'$ and between PE and PB', each of which is const. Hence the condition that the pedals shall be $\parallel$ is independent of the position of P. Take then P at A'. Then M', N' are at A'; hence $L'M'N'$ is the altitude $A'D'$, i. e. is $\perp B'C'$, i. e. $\perp BC$. Now $A'D$ bisects the angle EDF externally, i. e. bisects the angle MDN internally; hence MN is also $\perp BC$; i. e. the pedal $|^s$ are $\parallel$.

Ex. 62. $AA'/A'B = (\frac{1}{2} FA \,.\, FA' \sin AFA')$
$/(\frac{1}{2} FA' \,.\, FB \sin A'FB) = (FA/FB) \,.\, (\sin AFA'/\sin A'FB)$.
$CC'/C'D = C'C/DC' = (FC/FD) \,.\, (\sin A'FB/\sin AFA')$.

Hence $(AA'.CC')/(A'B.C'D) = (FA.FC)/(FB.FD)$.
So $(BB'.DD')/(B'C.D'A) = (EB.ED)/(EC.EA)$.
Hence we have to prove that

$$FA.FC.EB.ED = FB.FD.EC.EA.$$

But $FA/FB = \sin B/\sin A$, and so on. Hence the result.

Ex. 63. $KL:BC = KL:GH = OD:OA$.
Hence the product $\div AF.BD.CE$

$$= \frac{BC.OD}{BD.OA}\cdot\frac{CA.OE}{CE.OB}\cdot\frac{AB.OF}{AF.OC}$$

$$= \frac{(BOC).(OBD)}{(BOD).(OBA)}\cdot\frac{(OCA).(OCE)}{(OCE).(OBC)}\cdot\frac{(OAB).(OAF)}{(OAF).(OAC)} = 1.$$

Ex. 64. I is the o. c. of $I_1I_2I_3$. Hence BC is antiparallel to I_2I_3 w. r. to I_1I_2, I_1I_3. Hence I_1A', I_2B', I_3C' meet at the symmedian pt. of $I_1I_2I_3$.

Ex. 65. Call the sides and their reflexions a, b, c and a', b', c'. Then AD, BE, $CF \parallel a'$, b', c'. Hence

$$\sin ACF.\sin BAD.\sin CBE = \sin FCB.\sin DAC.\sin EBA.$$

For c', a are the reflexions of c, a' and hence

$$\angle FCB = \widehat{c'a} = \widehat{ca'} = BAD;$$

and so on.

Ex. 66. Project OA to infinity. Then in the new figure A is at infinity, $\therefore$ $CB = BD$. Also O is at infinity, $\therefore$ $CC' \parallel BB'$; and A'' is at infinity, $\therefore$ $BC' \parallel B''B'$. Now, as in the proof in p. 56, § 3, the value of $(AB''.CB)/(AC.BB'')$ is unaltered by proj^n. And in the new figure

$$(AB''.CB)/(AC.BB'')$$
$$= CB/BB'' = (CB/C'B').(C'B'/BB'')$$
$$= (BD/B'D).(B'D/B''D) = BD/B''D = C'D/B'D$$
$$= CD/BD = 2\text{; for as on p. 39, } AB''/AC = 1.$$

Ex. 67. A is a c. of s. of the two $\odot^s$. Hence the $\odot^s$ are homothetic w. r. to A. Consider the pt. Y homothetic to the given pt. Y', calling the given pt. Y' instead of Y for clearness. Then the tangents at Y, Y' are $\parallel$. Hence the tangents at X, Y to the incircle are $\parallel$, i. e. XY is a diameter of the incircle and hence is $\perp$ XY'. But Y is

on AY'; hence XY is known. Hence we can construct the incircle. Then AB, AC are the tangents from A.

Ex. 68. Take AO in any position. With centre A draw the $\odot^s$ g, h with radii AG, AH. With c. of s. O and ratio $3:1$ describe the $\odot$ h' homothetic with g; then since $OH = 3\,OG$, H is on h'. Hence H is an int^n of h, h' and is $\therefore$ known. Then $OA' \parallel AH$ and $= \frac{1}{2} AH$ gives A'. Then $BC \perp OA'$ gives the direction of BC. Then $OB = OC = OA$ gives B, C.

Ex. 69. Consider the centroid of masses m, $2m$, m at A, B, C. Then m, m at A, C can be replaced by $2m$ at Y; hence the centroid lies on BY. Also m, $2m$ at A, B can be replaced by $3m$ at Z; hence the centroid lies on CZ. Hence the centroid is P. Again consider the centroid of masses n, $2n$, $2n$ at A, B, C. Then $2n$, $2n$ at B, C can be replaced by $4n$ at X, and n, $2n$ at A, B can be replaced by $3n$ at Z. Hence the centroid lies on AX and CZ and is $\therefore$ at Q. Hence $3m\,.\,ZP = m\,.\,PC$ and $3n\,.\,ZQ = 2n\,.\,QC$, $\therefore$ $CP = \frac{3}{4} CZ$ and $CQ = \frac{3}{5} CZ$, $\therefore$ $CQ/QP = \frac{3}{5}/(\frac{3}{4} - \frac{3}{5}) = 4$. Also if AP cuts BC at T, $2m\,.\,BT = m\,.\,CT$. Lastly consider the centroid of $4m$, m, $2m$ at P, C, B. It is on BQ since $4m\,.\,PQ = m\,.\,CQ$, and on PT since $2m\,.\,BT = m\,.\,CT$. Hence it is R. Hence $4m\,.\,PS = 2m\,.\,BS$, $\therefore$ $PS = \frac{1}{3} BP$. But $2m\,.\,BP = 2m\,.\,PY$, $\therefore$ $PS = \frac{1}{6} BY$.

Ex. 70. Since $AZHY$ is a rectangle, the centre X of AH is on YZ. Hence it is enough to prove that the $\odot$ with centre X and radius XA cuts $A'X$ in pts. Y, Z which lie on the bisectors of A. Now $OA \parallel A'X$ (p. 24), $\therefore$ $\angle OAY = AYX = HAY$. Hence AY bisects OAH and $\therefore$ BAC (p. 47); also $HY \perp AY$. Now $AZ \perp AY$ and $HZ \perp AZ$. Hence HY, HZ are $\perp$ to the bisectors of A.

Ex. 71. Let AA', BB', CC' meet at O and BC, $B'C'$ meet at L and CA, $C'A'$ at M and AB, $A'B'$ at N; then L, M, N are collinear. Consider the Δ^s $CC'Y$, $BB'Z$. Since (CC'; BB'), ($C'Y$; $B'Z$), (YC; ZB) (viz. O, A, A') are collinear, the Δ^s are coaxal and $\therefore$ copolar, i. e. CB, $C'B'$, YZ concur. But CB, $C'B'$ meet at L. Hence BC, YZ meet at L. So

CA, ZX meet at M and AB, XY at N. But L, M, N are collinear. Hence the triangles ABC, XYZ are coaxal and $\therefore$ copolar.

Ex. 72. Project $X'YZ$ to infinity. Then in the new figure QZ', $Y'R$, BXC are $\parallel$ and also QB, $Z'X$, AC and also AB, $Y'X$, RC. Let QY' cut BC at S. Since $QZ' \parallel Y'R$, $\therefore$ $Z'R$ passes through S if $QZ'/Y'R = QS/Y'S$, i. e. if $BX/XC = QB/Y'C$, i. e. if $BX/XC = Z'X/Y'C$; and this is true by the similar Δ^s $BZ'X$, $XY'C$.

Ex. 73. The Δ^s ABC, PQR are copolar and $\therefore$ coaxal. Hence P', Q', R' are collinear. Hence PP', QQ', RR' are the diagonals of the quadrilateral $QRQ'R'Q$. Now see p. 90.

Ex. 74. $BX/XC = ZX/YC$ (by Δ^s BXZ, XCY) $= AY/YC$. Also from the pt. O and the $\Delta\, AZY$,

$(ZU/UY).(YC/AC).(AB/ZB) = 1$,

$$\therefore\ ZU/UY = (AC.ZB)/(YC.AB).$$

Hence we have to prove that $AY/YC = (AC.ZB)/(YC.AB)$ or $AY.AB = AC.ZB$ or $AB/AC = ZB/ZX$; which is true from the Δ^s BAC, BZX.

Ex. 75. Invert w. r. to A. Then A, B', C', D' are concyclic. The $\odot$ through A, $B \perp \odot\ ACD$ becomes the $|$ through $B' \perp |\ C'D'$; and so on. But these $\perp^s$ meet at H', the o. c. of $B'C'D'$. Hence in the given figure the three $\odot^s$ meet in a pt. H. Again let the centres and diameters of the $\odot^s$ ACD, ABD, ABC be P, Q, R, AL, AM, AN. Then $C'D'$ is the $\perp$ through L' to AP, and so on. Hence L', M', N' lie on the pedal $|$ of A w. r. to $B'C'D'$. Hence L, M, N lie on a $\odot$ through A. Also $AP:AQ:AR::AL:AM:AN$. Hence P, Q, R also lie on a $\odot$ through A. Bisect AH' at X', then $L'M'N'$ passes through X'. Hence the $\odot\ LMN$ passes through X on AH. Also $AX.AX' = AH.AH'$ and $AX' = \frac{1}{2}AH'$, $\therefore$ $AH = \frac{1}{2}AX$. But X lies on the $\odot$ LMN; hence H lies on the $\odot\ PQR$.

Ex. 76. Let the pairs of $|^s$ through A and B and C be a, a' and b, b' and c, c'. As one case, let a, b' cut at Z and a', b at Z' and c, a at Q and c', a' at Q' and b, c' at X and b', c at X'. Then we have to prove that XX', QQ', ZZ'

concur, i. e. that the Δ^s $XQ'Z'$, $X'QZ$ are copolar, i. e. that they are coaxal, i. e. that $(XQ'; X'Q)$, $(Q'Z'; QZ)$, $(Z'X; ZX')$ are collinear, i. e. that C, A, B are collinear; and this is true. Hence XX', QQ', ZZ' concur. Again if a, b cut at R and a', b' at R' and c, a' at Y and c', a at Y' and b, c at P and b', c' at P', we can, as above, prove that the six lines PP', QQ', RR', XX', YY', ZZ' meet three by three in four pts. and $\therefore$ form the six sides of a complete quadrangle.

Ex. 77. Let DE be the locus of A, E being on BC. Take F, the reflexion of C in DE. Then

$$\begin{aligned}\angle FAB &= FAE+EAB = CAE+EAB\\ &= C-E+180^\circ-B-E = 180^\circ+C-B-2E\end{aligned}$$

which is known. And A lies on the arc on BF containing this angle and also on DE and is $\therefore$ known.

Ex. 78. Let the tangent at P cut AB at X and the tangent at A at the pt. Y. Let QM be the $\perp$ from Q on AB. Take O the centre of the $\odot$. Draw $PN \perp AB$ and $PL \perp AY$. Then by the Δ^s AQM, OPN we have $QM/PN = AQ/OP$. Hence $QM \propto PN.AQ$. Now $\angle QPA = 90^\circ - APO = 90^\circ - PAO = NPA$ and $AP = AP$ and $Q = N = 90^\circ$. Hence the Δ^s QPA, NPA are congruent, $\therefore AQ = AN = PL$. Hence we want $PN.PL$ greatest; i. e. area $PLAN$ greatest. Now take a consecutive position $P'L'AN'$. Then area $PLAN$ = area $P'L'AN'$. Hence area $PNN'Z$ = area $PL'LZ$ if PL and $P'N'$ cut at Z; i. e. $PN.PZ = P'L'.P'Z$, or $PN/P'L' = P'Z:PZ$, or ult^{ly} $PN/PL = PN/NX$, $\therefore PL = NX$, $\therefore PY = PX$. Greatest because a unique critical value between two zero values; viz. when P is at A or B.

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